## THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred fourteenth regular meeting of the Society was held at Columbia University on Saturday, February 26, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members:

Professor J. W. Alexander, Mr. D. R. Belcher, Professor A. A. Bennett, Professor E. W. Brown, Dr. G. A. Campbell, Dr. Tobias Dantzig, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor H. B. Fine, Mr. R. M. Foster, Mr. Philip Franklin, Dr. T. C. Fry, Professor O. E. Glenn, Dr. T. H. Gronwall, Dr. C. C. Grove, Dr. A. A. Himwich, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Dr. E. A. T. Kircher, Dr. K. W. Lamson, Mr. Harry Langman, Professor G. W. Mullins, Professor F. W. Owens, Dr. E. L. Post, Professor H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Miss Louise E. C. Stuerm, Professor H. D. Thompson, Professor E. B. Wilson.

Ex-President H. B. Fine occupied the chair. The Council announced the election of the following one hundred fourteen persons to membership in the Society:

Professor Orrin Wilson Albert, Grinnell College;
Professor David Robert Allen, University of Utah;
Mr. Elbert Frank Allen, University of Missouri;
Professor Charles Lincoln Arnold, Ohio State University;
Miss Mary Caroline Ball, Northwestern University;
Professor Wightman Samuel Beckwith, Ohio Northern University;
Mr. William Noël Birchby, California Institute of Technology;
Dr. Edwin Mortimer Blake, Brooklyn, N. Y.;
Miss Rachel Blodgett, Harvard University;
Professor Rosser Daniel Bohannan, Ohio State University;
Professor John David Bond, Agricultural and Mechanical College of Texas;
Mr. Frederick William Borgward, Syracuse University;
Dr. Henry Roy Brahana, University of Illinois;
Professor William Mayo Brodie, Virginia Polytechnic Institute;
Mr. Bancroft Huntington Brown, Harvard University;
Professor Lillian Olive Brown, Hood College;
Professor Frank Newton Bryant, Syracuse University;
Miss Minnie Wilford Caldwell, Hardin College;
Miss Evelyn Teresa Carroll, Wells College;

Mr. Ming-Cheng Chow, University of Cincinnati;
Mr. Oliver Charles Collins, University of Nebraska;
Professor Allan Ray Congdon, University of Nebraska;
Miss Mary Lucile Copenhover, University of Oregon;
Miss Julia Dale, University of Missouri;
Mr. Harold Thayer Davis, University of Wisconsin;
Miss Alice Crowell Dean, Rice Institute;
Professor Dewey Stevens Dearman, Millsaps Academy;
Professor Alexander Dillingham, United States Naval Academy;
Professor Eleanor Catherine Doak, Mount Holyoke College;
Mr. Theodore Doll, Northwestern University;
Rev. Joseph Nicholas Donohue, Notre Dame University;
Mr. Finis Omer Duncan, University of Missouri;
Miss Edna May Feltges, University of Wisconsin;
Mr. Malcolm Cecil Foster, Yale University;
Mr. Ronald Martin Foster, American Telephone and Telegraph Company;
Mr. Percy Austin Fraleigh, Cornell University;
Mr. Bennington Pearson Gill, College of the City of New York;
Mr. Rutherford Erwin Gleason, State University of Iowa;
Miss Neoma Lillian Goldsberry, University of Missouri;
Professor Cornelius Gouwens, Iowa State College;
Professor Harry Emil Gudheim, Virginia Polytechnic Institute;
Professor Truman Leigh Hamlin, Clarkson College;
Mr. Robert William Hartley, University of Pennsylvania;
Miss Camilla Hayden, University of Wisconsin;
Dr. Henry Benjamin Hedrick, United States Army Ordnance;
Miss Gertrude Anne Herr, Iowa State College;
Dr. Carl Einar Hille, University of Stockholm;
Professor Allan Wilson Hobbs, University of North Carolina;
Miss Clarice Sarah Hobensack, Ohio State University;
Miss Helma Lou Holmes, University of Texas;
Professor Alfred Hume, University of Mississippi;
Mr. William Anderson Hutcheson, Mutual Life Insurance Company;
Mr. Mark Hoyt Ingraham, University of Wisconsin;
Miss Margaret Eloise Jones, Ohio State University;
Professor Frederick Albert Lewis, University of Alabama;
Dr. Peysah Leyzerah, Lehigh University;
Professor Louis Lindsey, Syracuse University;
Mr. Joe Burton Linker, University of North Carolina;
Mr. Ralph Alden Loring, Dartmouth College;
Mr. Harold Marshall Lufkin, Cornell University;
Professor Edward Hiram McAlister, University of Oregon;
Miss Mary Bell McMillan, Wisconsin State Normal School, River Falls;
Mr. Israel Maizlish, Reed College;
Mr. Edward Lawrence Milne, Lake Forest Academy;
Mr. Thessalon Herbert Milne, University of Alberta;
Mr. Gordon Richmond Mirick, University of Rochester;
Mr. Allen Guy Montgomery, University of West Virginia;
Mr. David Sherman Morse, Cornell University;
Mr. Ray Dickinson Murphy, Equitable Life Assurance Society;
Dr. Almar Naess, Naval Academy of Norway;
Dr. Jason John Nassau, Syracuse University;
Mr. Charles Edward Norwood, United States Army Ordnance;
Professor John Hutcheson Ogburn, Lehigh University;
Professor Earnest Jackson Oglesby, New York University;
Mr. Jesse Otto Osborn, Pennsylvania State College;
Mr. Leroy Elden Peabody, Yale University;
Miss Elsie Marie Plapp, University of Wisconsin;

Mr. Hillel Poritsky, Cornell University;
Miss Jessie Grace Quigley, University of Missouri;
Mr. Ottis Howard Rechard, University of Wisconsin;
Professor Jacob Charles Rietz, Ohio State University;
Professor Lulu Runge, University of Nebraska;
Mr. Howard Conway Shaub, Dartmouth College;
Mr. Charles Robert Sherer, University of Nebraska;
Professor Thomas McNider Simpson, Jr., Randolph-Macon College;
Miss Helen Florene Smith, Iowa State College;
Professor Merlin Grant Smith, Greenville College;
Mr. Marvin Reinhard Solt, Lehigh University;
Professor Charles Stillman Sperry, University of Colorado;
Mr. Eugene Stephens, Washington University;
Mrs. Elizabeth Easton Stickney, University of Illinois;
Miss Louise Elizabeth Cathrine Stuerm, American Telephone and Telegraph Company;
Professor Mary Clegg Suffa, Elmira College;
Mr. Frank Elmer Swift, Notre Dame University ;
Mr. James Henry Taylor, University of Nebraska;
Mr. Van B. Teach, Ohio State University;
Mr. Joseph Miller Thomas, University of Pennsylvania;
Professor Perley Lenwood Thorne, New York University;
Professor Joseph Ellis Trevor, Cornell University;
Professor Joseph Henry Tudor, Pennsylvania State College;
Professor Horace Scudder Uhler, Yale University;
Professor Harry Clark Van Buskirk, California Institute of Technology;
Mr. Charles Conroy Wagner, Pennsylvania State College;
Mr. Lewis Edes Ward, Harvard University;
Professor Emery Ernest Watson, Iowa State College for Teachers;
Professor Warren Weaver, University of Wisconsin;
Professor Ross Albert Wells, Michigan State Normal College;
Mr. Karl Leland Wildes, Massachusetts Institute of Technology;
Professor Robert Daniel Williams, Ohio State University;
Professor Charles Owen Williamson, College of Wooster;
Professor Harold Albert Wilson, Rice Institute;
Mr. Frederick Wood, University of Wisconsin;
Dr. Roscoe Woods, State University of Iowa;
Miss Jessica May Young, Washington University.

Twenty-four applications for membership in the Society were received.

The Council voted to accept the invitation extended by the American Association for the Advancement of Science to affiliate with it. Professor E. B. Van Vleck was appointed a representative of the Society in the division of physical sciences of the National Research Council, as successor to Professor H. S. White, whose term will soon expire.

The final report from Professor E. R. Hedrick, as chairman of the Committee on membership and sales, was received, and the committee was discharged at its own request, with the thanks of the Council for its notable services at this critical time. In all one hundred thirty-two applications for membership and seventy-seven subscriptions to the TRANSACTIONS for the current year have been received through this committee.

Questions of policy were raised concerning dues for foreign members, concerning choice between BULLETIN and TRANSACTIONS in the case of foreign members, concerning sales and exchanges of publications with foreign societies and libraries, and concerning individual or concerted efforts to aid foreign journals. It was voted to authorize the President to appoint a committee to consider these and related questions.

The secretary was requested to write on behalf of the Council a letter of felicitation to Professor Magnus Gösta Mittag-Leffler on the occasion of the seventy-fifth anniversary of his birth, March 16, 1921.

A letter from Professor F. N. Cole was read to the Council, donating to the Society the sum which accompanied the testimonial tendered him at the preceding meeting of the Society in recognition of his very distinguished services. It was voted that the Council accept the gift and extend to Professor Cole its heartiest appreciation of his generosity; it was further voted that this fund shall constitute and be designated as the Cole Fund, and the President was requested to appoint a committee to consider the use to which the income can best be devoted. The Council approved the suggestion that the present volume of the BULLETIN be inscribed to Professor Cole.

The papers read at the February meeting are stated below. Those of Mr. Rice, Professor Glenn, Professor Moore, Dr. Zeldin, and Miss Mullikin, and the second paper of Dr. Ritt, were read by title.

1. Mr. L. H. Rice: *Coefficient of the general term in the expansion of a product of polynomials.*

The writer shows, first, that in the expansion of the product $(a_1 + a_2 + \cdots + a_m)(a_2 + \cdots + a_m)\cdots(a_{m-1} + a_m)a_m$ the coefficient of the general term $Aa_{a_1}^{\beta_1}a_{a_2}^{\beta_2}\cdots a_{a_{f-1}}^{\beta_{f-1}}a_m^{\beta_f}$, where $\alpha_1 < \alpha_2 < \cdots < m$, is:

$$A = \binom{\alpha_1}{\beta_1}\binom{\alpha_2 - \beta_1}{\beta_2}\binom{\alpha_3 - \beta_2 - \beta_1}{\beta_3} \cdots \binom{\alpha_{f-1} - \beta_{f-2} - \cdots - \beta_1}{\beta_{f-1}}.$$

More generally, in the expansion of the product

$$(a_{t_1} + a_{t_1+1} + \cdots + a_n)(a_{t_2} + a_{t_2+1} + \cdots + a_n) \cdots (a_{t_m} + a_{t_m+1} + \cdots + a_n)$$

the coefficient of the general term $Ba_{a_1}^{\beta_1} a_{a_2}^{\beta_2} \cdots a_{a_f}^{\beta_f}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_f$, is

$$B = \binom{\alpha_1'}{\beta_1}\binom{\alpha_2' - \beta_1}{\beta_2}\binom{\alpha_3' - \beta_2 - \beta_1}{\beta_3} \cdots \binom{\alpha'_{f-1} - \beta_{f-2} - \cdots - \beta_1}{\beta_{f-1}},$$

where $\alpha_j'$ is the number of $\alpha_j$'s in the factors of the given product.

2. Professor E. V. Huntington: *The mathematical theory of proportional representation, with a substitute for least squares.*

The author shows that the new method of apportionment which he presented at the December meeting may be based on the following categorical set of postulates (in which $A$, $B$, $C$, $\cdots$ are the populations of the states, and $a$, $b$, $c$, $\cdots$ the number of representatives assigned to each): I. For the case of two states, the ratio between $A/a$ and $B/b$ (or $a/A$ and $b/B$; or $A/B$ and $a/b$; or $B/A$ and $b/a$) should be as near unity as possible. II. No pair of states should be capable of being *improved* by a transfer of representatives within that pair.

3. Professor F. W. Owens: *On the apportionment of representatives.*

This paper compares various used or proposed methods of apportionment, pointing out what might be considered as advantages and disadvantages, with illustrative examples. It is also shown that a justification of the method of major fractions may be found in a form of least squares. Suggestions are also offered as to special treatment of the first representative from each state.

4. Professor Arnold Emch: *On the polar equation of algebraic curves.*

The author studies the transformation

(1) $\sigma x_1' = x_2(x_3^2 - x_1^2)$, $\sigma x_2' = 2x_1x_2x_3$, $\sigma x_3' = (x_1^2 + x_3^2)x_3$

between two planes $(x')$ and $(x)$, which occurs in the change

from polar to cartesian coordinates. If we set $x_1/x_3 = t = \tan(\theta/2)$, $x_2/x_3 = \rho$, $x_1'/x_3' = x$, $x_2'/x_3' = y$, the polar equation of an algebraic curve $F(\rho, \cos\theta, \sin\theta) = 0$ may be written in the form $\phi^*(t, \rho) = 0$, or $\phi(x_1, x_2, x_3) = 0$. By means of the transformation (1) the equation $\phi = 0$ is transformed into the projective equation $f(x_1', x_2', x_3') = 0$ of the algebraic curve $F = 0$. Interpreting $\phi$ in the projective plane $(x)$, we may speak of the $\phi$-curve. The fundamental points of the transformation (1) are the double point (0, 1, 0), and the single points (1, 0, 0), (1, 0, $i$), (1, 0, $-i$).

It is shown that to the polar equation in the $\phi$-form of order $n$ and deficiency $p$, in a general position, i.e., independent of the fundamental points, corresponds an algebraic curve $f$ of order $3n$ and the same deficiency. The singularities of $f$ are discussed in detail and are described in full.

Reductions in the order of $f$ and in its multiple points due to the passage of $\phi$ through the fundamental points or the arrangement of the points of $\phi$ in couples are also considered. The method outlined in this paper is useful in studying the singularities of an algebraic curve from its polar equation.

5. Professor O. E. Glenn: *Generalization of the concept of invariancy derived from a type of correspondence between functional domains. Second proof of the finiteness of formal binary concomitants modulo p.*

Two functional domains $D$, $Q$ may be so related that, by a transformation, there exists, for functions $\varphi$ of $D$, an invariant relation $\varphi' = r\varphi$ while a correspondence $\varphi \sim \omega$ with functions $\omega$ of $Q$ holds which implies also $\omega' = s\omega$. There follows $\varphi = q\omega$ where $\varphi$, in some particular cases, is the same function of its arguments as is $\omega$, and in other cases a different function. Combining the above with a theory of invariant elements, the author obtains a proof of the finiteness of formal modular concomitant systems in better alignment with formal methods than his proof given at the December (1920) meeting.

6. Professor R. L. Moore: *Concerning the sum of a countable number of closed point sets.*

In the Tôhoku Mathematical Journal, vol. 13, June, 1918, pp. 300–303, Sierpinski shows that, in a space of $m$ dimensions, if the sum $M$ of a countable number of mutually exclusive closed point sets is closed and bounded, then $M$ is

not connected. He points out that if the stipulation that $M$ be bounded is removed the thus modified proposition holds for $m = 1$ but not for $m = 3$. He raises the question whether it holds for $m = 2$. Professor Moore answers this question by showing the existence of an unbounded closed and connected plane point set which is the sum of a countable number of mutually exclusive closed point sets. He shows, in addition, that Sierpinski's theorem remains true for every $m$ if the stipulation that $M$ be bounded is retained but the stipulation that $M$ be closed is replaced by the stipulation that if there be any limit point of $M$ that does not belong to $M$ then the set of all such limit points be closed.

7. Dr. S. D. Zeldin: *On the simplification of the structure of finite continuous groups with more than one two-parameter invariant subgroup.*

In this paper the author shows how the structure of a group of order $r + 2k$ with $k > 1$ two-parameter invariant subgroups can be simplified if its meroedrically isomorphic group of order $r$ has one invariant spread. He shows that the adjoint of the group $G_{r+2k}$ has $2k + 1$ invariants, one of which is the invariant of the adjoint of the isomorphic group $G_r$, and by considering the different possible forms of these invariants he finds some of the corresponding structural constants.

8. Dr. J. F. Ritt: *Periodic functions with a multiplication theorem.*

Let $f(z)$ represent a uniform analytic function with no essential singularity in the finite part of the plane, and periodic, either simply or doubly. Dr. Ritt determines all cases in which a number $m$ exists such that $f(mz)$ is a rational function of $f(z)$. The results are practically negative. The modulus of $m$ cannot be less than unity. If $|m| > 1$ and if $f(z)$ is simply periodic, $f(z)$ is a linear function of $e^{\alpha z}$ or of $\cos(\alpha z + \beta)$, where $\beta$ is restricted to certain values. If $|m| > 1$ and if $f(z)$ is doubly periodic, $f(z)$ must be a linear function of $p(z + \beta)$, except in the lemniscatic and equianharmonic cases. In the lemniscatic case $f(z)$ may be a linear function of $p^2(z + \beta)$ and in the equianharmonic case it may be a linear function either of $p^3(z + \beta)$ or of $p'(z + \beta)$. For $|m| = 1$, there is a wider variety of possibilities.

9. Dr. J. F. Ritt: *Note on equal continuity.*

This paper appears in the present number of this BULLETIN.

10. Professor I. J. Schwatt: *Expressions for the Bernoulli function of order p.*

Eisenlohr, Schlömilch, Worpitzky and others have derived expressions for the Bernoulli function. But their methods assume expansions involving the Bernoulli numbers. The author obtains expressions for the Bernoulli function directly from its definition.

11. Professor I. J. Schwatt: *The expansion of a continued product.*

By means of certain principles of operation with series, the author obtains the expansion of $\Pi_{k=1}^{p} \sin kx$ and also of $\sum_{n=0}^{p} \sum_{k=1}^{n-1} \sin^t (k\pi/n)$ and of similar summations of continued products.

12. Professor I. J. Schwatt: *Method for the summation of a family of series.*

A further extension of the case which the author treated in the NOUVELLES ANNALES DE MATHÉMATIQUES, (4), vol. 16, May, 1916, pp. 203–209, is considered in this paper.

13. Professor I. J. Schwatt: *Note on the evaluation of a definite integral.*

The evaluation of a definite integral from its definition as a summation presents even for some of the simpler cases considerable difficulty. The integrals of expressions which the author is unable to reduce to differentials of known functions lead to series which he cannot sum. Some of the principles involved in the summation of series obtained during the process of evaluating the definite integrals considered in this paper are believed to be new.

14. Mr. John McDonnell: *A property of the Pellian equation with some results derived from it.*

This paper shows that if $(t, u)$ be any solution of the Pellian equation

$$t^2 - Du^2 = 1,$$

then

$$t + u\sqrt{D} = (x\sqrt{M} + y\sqrt{N})^2,$$

where $(x, y)$ is a solution of one of the equations $Mx^2 - Ny^2 = 1$ or $Mx^2 - Ny^2 = 2$, and $MN = D$, where either $M$ or $N$ may be unity. From this result certain conclusions are drawn regarding the fundamental solution of the Pellian equation. It is shown as a further consequence that certain theorems on prime numbers established by Lucas in 1878 may be stated with greater precision, and the criteria derived from them rendered more definite.

15. Miss Anna M. Mullikin (introduced by Professor R. L. Moore): *A necessary and sufficient condition that the sum of two bounded, closed and connected point sets should disconnect the plane.*

The author shows that if, in a space $S$ of two dimensions, $m$ and $n$ are two bounded, closed, and connected point sets, neither of which separates $S$, then (1) in order that the point set $m + n$ should separate $S$ into at least two parts, it is necessary and sufficient that the set of points common to $m$ and $n$ should not be connected, (2) in order that $m + n$ should separate $S$ into just two connected domains it is sufficient that the set of points common to $m$ and $n$ should be the sum of two closed connected point sets that have no point in common.

16. Dr. T. H. Gronwall: *Some empirical formulas in ballistics.*

The author deduces approximate formulas for the range variations due to a change in temperature and to the rotation of the earth in terms of the variations due to changes in initial velocity, angle of departure and ballistic coefficient, thus making it possible to obtain all the differential variations in range by simple processes of interpolation from the ballistic tables devised by Professor Bennett and now being computed under his direction.

17. Dr. T. H. Gronwall: *Summation of a double series.*

In this paper, it is shown, by the use of Cauchy's theorem, that the double series

$$\sum_{m,n=1}^{\infty} \frac{(m+n-2)!\,(m+n-1)!}{m!\,(m-1)!\,n!\,(n-1)!}\, x^{2m}y^{2n},$$

which was encountered by Dr. K. W. Lamson in the solution

of a question in mathematical physics, has the sum

$$\tfrac{1}{2}[1 - x^2 - y^2 - \sqrt{(1+x+y)(1+x-y)(1-x+y)(1-x-y)}],$$

the domain of convergence being $|x| + |y| < 1$.

18. Professor L. P. Eisenhart: *A geometric characterization of the paths of particles in the gravitational field of a mass at rest.*

Schwarzschild integrated the differential equations of a permanent gravitational field in the Einstein theory and obtained the quadratic form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 + \frac{r}{2m-r}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The geodesics of the four-space with this linear element are the paths of a particle moving in the gravitational field produced by a mass $m$ at rest at the point $r = 0$. It can be shown that these geodesics can be identified with the geodesics of a three-space $\Sigma$ with the linear element

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 + \frac{r}{2m-r}dr^2 - r^2d\theta^2.$$

The author has shown that $\Sigma$ is characterized by the following properties: (i) it admits a continuous group $G_2$ of motions into itself; (ii) the $\infty^1$ surfaces $S_1$ of $\Sigma$ which move into themselves for each motion of the group form part of a triply orthogonal system; (iii) the directions of the curves of intersection of the surfaces, $S_1, S_2, S_3$, of the triple system through a point $P$ are the principal directions of $\Sigma$ at $P$, in the sense that one of the surfaces, $S_1, S_2, S_3$, through $P$ has the maximum gaussian curvature and another the minimum curvature at $P$ of all the surfaces of $\Sigma$ through $P$; (iv) the curvature of $S_1$ is different from zero and equal to the curvature of $S_2$ or $S_3$, and the curvature of the other is minus twice that of $S_1$. This paper will appear in the ANNALS OF MATHEMATICS.

19. Professor A. A. Bennett: *The equations of interior ballistics.*

In this paper, the derivation and interpretation of a set of three fundamental equations for interior ballistics are dis-

cussed. These equations are in approximately the usual form, except for a change of variables. This change of variables is necessitated if tables are to be constructed to give all of the data desired for interior ballistics. The equations have been integrated numerically in the office of the Technical Staff of the Ordnance Department, and tables have been constructed which will shortly be published as a Government Document. The novelty of the undertaking consists in the introduction for the first time of new standard variables, which have a fundamental physical significance. The tables secured are independent of physical dimensions and appear to be the first to be published which correspond directly to the equations in their classical form. The results from the tables must be multiplied by external factors which involve the physical dimensions and depend upon the units chosen.

R. G. D. RICHARDSON,
*Secretary.*

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## NOTE ON EQUAL CONTINUITY.

BY DR. J. F. RITT.

(Read before the American Mathematical Society February 26, 1921.)

The notion of equal continuity, introduced by Ascoli, has acquired prominence through applications made of it by Hilbert, Montel and others.* A family of functions, defined on an interval, is equally continuous at a point if for every positive number there is an interval containing the point, in which the oscillation of each of the functions is less than the number.

Dealing with families of functions which are not equally continuous, we shall record here some properties of a function which will be called the *saltus* of the family.

Choosing a point on the interval of definition of the functions, let us suppose that there exists a number for which an interval can be found, containing the point, in which the oscillation of each of the functions is less than the number.

* For the literature on this question, see Montel, *Sur les suites infinies de fonctions*, ANNALES DE L'ECOLE NORMALE, 1907.

The greatest lower bound of all such numbers will be called the *saltus* of the family at the point. If no such number exists, the saltus will be taken as $+\infty$. Of course, wherever the saltus vanishes, the family is equally continuous.

The saltus is an upper semi-continuous function on the interval considered. Consequently, if it vanishes nowhere in an interval, there is a sub-interval within that interval in which its lower bound exceeds zero.*

We shall apply the notion of the saltus of a family of functions to the case of a convergent sequence of continuous functions. Let the sequence of functions

$$\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots,$$

each continuous in the interval $(a, b)$,† converge throughout that interval. It is well known that wherever the sequence is equally continuous, the limit of the sequence is continuous.

Suppose that there be an interval $(\alpha, \beta)$ for no point of which the saltus of the sequence is zero. There must be an interval $(\alpha_1, \beta_1)$, within $(\alpha, \beta)$, for every point of which the saltus exceeds some positive number $\epsilon$. Let $\varphi_{i_1}(x)$ be any function of the above sequence, and $x_0$ any point of $(\alpha_1, \beta_1)$. In a small neighborhood of $x_0$, the oscillation of $\varphi_{i_1}(x)$ is very small. But for any such small neighborhood there is a function $\varphi_{i_2}(x)$, with $i_2 > i_1$, whose oscillation in that neighborhood exceeds $\epsilon$. Hence there is an interval $(\alpha_2, \beta_2)$, interior to $(\alpha_1, \beta_1)$, in which $\varphi_{i_2}(x)$ differs from $\varphi_{i_1}(x)$ by more than $\epsilon/3$. Continuing thus, we have a sequence of intervals

$$(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_n, \beta_n), \cdots,$$

each interior to the preceding one, and a sequence of functions

$$\varphi_{i_1}(x), \varphi_{i_2}(x), \cdots, \varphi_{i_n}(x), \cdots,$$

with $i_n > i_{n-1}$, such that

$$|\varphi_{i_n}(x) - \varphi_{i_{n-1}}(x)| > \frac{\epsilon}{3}$$

throughout $(\alpha_n, \beta_n)$. It is clear that the original sequence could not converge at the points common to the intervals $(\alpha_n, \beta_n)$.

* Hobson, *Theory of Functions*, p. 238.

† It does not matter whether the interval is taken as open or as closed.

We find thus Baire's well known theorem to the effect that the limit of a sequence of continuous functions is at most point-wise discontinuous.

In connection with convergent sequences of continuous functions, the saltus function here considered can be related with the *measure of non-uniform convergence* introduced by Hobson and Osgood.* These two functions vanish at the same points, which fact shows, of course, that the above proof of Baire's theorem is not fundamentally distinct from that based on the measure of non-uniform convergence. There is no other relation of equality between the two functions.

COLUMBIA UNIVERSITY.

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# A NEW METHOD IN DIOPHANTINE ANALYSIS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society March 26, 1921.)

1. *Introduction and Summary.* In the preceding number of this BULLETIN (p. 312) I gave reasons why due caution should be observed toward the literature on the solution of homogeneous equations in integers. The valid knowledge concerning this subject is much less than has been usually admitted. The lack of general methods is even greater than in the subject of non-homogeneous equations. The chief aim of the present paper is to suggest such a method, based on the theory of ideals. The method is applicable in simple cases (§§ 2–4) without introducing ideals.

For the sake of brevity we shall restrict attention to the problem of finding all integral solutions of the equation

$$x_1^2 + ax_2^2 + bx_3^2 = x_4^2,$$

an equation admitted† to be difficult of treatment by any known methods, and previously solved completely in integers only in the single case $a = b = 1$.

Let us write

$$x_4 - x_1 = z, \quad x_4 + x_1 = w.$$

Then from the integral solutions of $ax^2 + by^2 = zw$ we must

* Hobson, loc. cit., p. 484.

† Carmichael, *Diophantine Analysis*, 1915, p. 38.

select those for which $z \equiv w \pmod 2$. The trouble in making such a (simple) selection is avoided when $a = 1$, $b = 4k - 1$, by a reduction* to

$$x^2 + xy + ky^2 = zw.$$

Further, any solution of

$$ax^2 + by^2 = zw$$

yields a solution $X = ax$, $Y = y$, $Z = az$, $W = w$, of $X^2 + abY^2 = ZW$ in which $X$ and $Z$ are divisible by $a$, and conversely. Hence our problem leads to the canonical equation $N = zw$, where $N$ is $x^2 - my^2$ or $x^2 + xy + ky^2$. We shall see that the theory of algebraic numbers is admirably adapted to the complete solution of $N = zw$ in integers.

2. *Definition of Integral Algebraic Numbers.* Let $m$ be an integer other than 0 and 1, and such that $m$ is not divisible by a perfect square. The numbers $\tau = r + s\sqrt{m}$, where $r$ and $s$ are rational, form a domain of rationality (or field) $R(\sqrt{m})$. Evidently $\tau$ and its *conjugate* $\tau' = r - s\sqrt{m}$ are the roots of the equation

$$x^2 - 2rx + r^2 - ms^2 = 0.$$

If the coefficients of this quadratic are all integers, $\tau$ and $\tau'$ are called *integral algebraic numbers* of $R(\sqrt{m})$. A simple discussion† leads to the following theorem.

THEOREM 1. *The integral algebraic numbers of $R(\sqrt{m})$ are $x + y\theta$, where $x$ and $y$ are integers, and where*

(1) $\qquad \theta = \sqrt{m}$ *if* $m \equiv 2$ *or* $m \equiv 3 \pmod 4$,

(1′) $\quad \theta = \frac{1}{2}(1 + \sqrt{m})$, $\theta^2 - \theta + \frac{1}{4}(1 - m) = 0$, *if* $m \equiv 1 \pmod 4$.

The conjugate to $\xi = x + y\theta$ is $\xi' = x + y\theta'$, where $\theta' = -\sqrt{m}$ in case (1), and $\theta' = \frac{1}{2}(1 - \sqrt{m})$ in case (1′). The product $\xi\xi'$ is called the *norm* of $\xi$ and denoted by $N(\xi)$. Hence, in the respective cases,

* Details are given in the writer's address before the International Mathematical Congress at Strasbourg, in which he described the present method for the simplest cases, including the solution, by use of the arithmetic of quaternions (see PROCEEDINGS LONDON MATH. SOC., 1921), of $x^2 + y^2 + z^2 + \eta^2 = zw$ and hence of $x_1^2 + \cdots + x_8^2 = x_9^2$.

† See this BULLETIN, vol. 13 (1906–7), p. 350, or any text on algebraic numbers.

(2) or (2′) $N(x+y\theta)=x^2-my^2$ or $x^2+xy+\frac{1}{4}(1-m)y^2$.

3. *All Rational Solutions of* $N(x+y\theta)=zw$. If $z\neq 0$, we may write

$$\frac{x}{z}=\frac{a}{c},\qquad \frac{y}{z}=\frac{b}{c},$$

where $a$, $b$, and $c$ are integers without a common factor greater than 1. Then we have

$$\frac{w}{z}=\frac{N(x+y\theta)}{z^2}=N\left(\frac{x}{z}+\frac{y\theta}{z}\right)=N\left(\frac{a+b\theta}{c}\right)=\frac{N(a+b\theta)}{c^2}.$$

If we take $\rho=z/c^2$, where $\rho$ is rational, we obtain

(3) $x=\rho ac,\qquad y=\rho bc,\qquad z=\rho c^2,\qquad w=\rho N(a+b\theta).$

But if $z=0$, then $N(x+y\theta)=0$ and $x=y=0$, and this solution is the case $c=0$ of (3).

THEOREM 2. *All rational solutions of* $N(x+y\theta)=zw$ *are given by* (3), *where* $a$, $b$, $c$ *are integers without a common factor and* $\rho$ *is rational.*

4. *Integral Solutions without the Use of Ideals.* Not all integral solutions of $N(x+y\theta)=zw$ are obtained from (3) by restricting $\rho$ to integral values, as was shown in § 4 of my preceding paper. To obtain further solutions, note that the norm of the product

(4) $$x+y\theta=(a+b\theta)(c+d\theta)$$

of two numbers of our field $R(\theta)$ equals the product of their norms. Hence $N(x+y\theta)=zw$ has the solution

(5) $z=N(c+d\theta),\qquad w=N(a+b\theta),\qquad x, y$ by (4);

or explicitly,

($5_1$) $x=ac+mbd,\qquad y=ad+bc,\qquad z=c^2-md^2,$

$$w=a^2-mb^2,\qquad m\equiv 2 \text{ or } 3 \pmod 4;$$

($5_2$) $x=ac+\dfrac{(m-1)}{4}bd,\qquad y=ad+bc+bd,$

$$z=c^2+cd+\tfrac{1}{4}(1-m)d^2,\qquad w=a^2+ab+\tfrac{1}{4}(1-m)b^2,$$

where $m\equiv 1 \pmod 4$.

We shall restrict attention to integral values of $a$, $b$, $c$, and $d$

without a common factor. The products of the resulting numbers (5) by an arbitrary rational number $\rho$ give all rational solutions of $N(x + y\theta) = zw$, since those products reduce to (3) when $d = 0$.

When the field $R(\theta)$ is such that its integral algebraic numbers $x + y\theta$ obey the laws of divisibility of arithmetic, we shall prove that all integral solutions of $N(x + y\theta) = zw$ are given by the products of the numbers (5) by an arbitrary integer $\rho$, where $a$, $b$, $c$, and $d$ are integers without a common factor. We have merely to show that when the products of the numbers (5) by an irreducible fraction $n/p$ are integers, so that the numbers (5) are divisible by $p$, then the quotients are expressible in the same form (5) with new integral parameters in place of $a$, $b$, $c$, and $d$. It suffices to prove this for the prime factors (equal or distinct) of $p$, since after each of them has been divided out in turn, $p$ itself has been divided out.

Let therefore $p$ be a prime which divides the four numbers (5). If $p$ divided both $d$ and $b$, it would divide also $c$ and $a$, in view of $z$ and $w$, contrary to the hypothesis that $a$, $b$, $c$, and $d$ have no common factor. By the interchange of $a$ with $c$ and $b$ with $d$, $x$ and $y$ remain unaltered, while $z$ and $w$ are interchanged. Hence we shall be treating one of two entirely similar cases if we assume that $d$ is not divisible by $p$.

The prime $p$ divides the product $z$ of $c + d\theta$ and $c + d\theta'$, without dividing either factor. For, if $c + d\theta$ or $c + d\theta' = c + d(\alpha - \theta)$, where $\alpha = 0$ or 1 in the respective cases (1) or (1′), were the product of $p$ by $k + l\theta$, where $k$ and $l$ are integers, then $\pm d = pl$, whereas $d$ is not divisible by $p$. Since the laws of divisibility of arithmetic were assumed to hold for the integral numbers of $R(\theta)$, it follows that $p$ is not an algebraic prime, but decomposes into $p = \pi\pi'$, where $\pi$ and $\pi'$ are conjugate primes.* By choice of the notation between $\pi$ and $\pi'$, we may assume that $\pi$ is the one of the two prime factors $\pi$ and $\pi'$ of $p$ which divides $c + d\theta$, and we may write

$$(6) \qquad c + d\theta = \pi(C + D\theta), \qquad z = pN(C + D\theta),$$

where $C$ and $D$ are integers. Since $x$ and $y$ are divisible by $p$,

* Otherwise, $p$ would be a product of three integral algebraic numbers, no one a unit, and its norm $p^2$ would be a product of three integers no one of which is $\pm 1$. A unit $u$ is an integral algebraic number which divides unity, whence $N(u) = \pm 1$ ($+1$ if $m$ is negative).

and $c + d\theta$ is divisible by $\pi$, but not by the product $p = \pi\pi'$, it follows from (4) that $a + b\theta$ is divisible by $\pi'$, i.e.

$$(7) \qquad a + b\theta = \pi'(A + B\theta), \qquad w = pN(A + B\theta),$$

where $A$ and $B$ are integers.

Comparing the product of (6) and (7) with (4), we have

$$x + y\theta = p(\xi + \eta\theta), \qquad \xi + \eta\theta \equiv (A + B\theta)(C + D\theta).$$

Hence the integral quotients $\xi = x/p$, $\eta = y/p$, $z/p$, $w/p$ are of the form (5) with $a$, $b$, $c$, and $d$ replaced by the integers $A$, $B$, $C$, and $D$. This proves the following theorem.

THEOREM 3. *All integral solutions of $N(x + y\theta) = zw$ are obtained by multiplying an arbitrary integer by the numbers* (5) *in which $a$, $b$, $c$, and $d$ are integers without a common factor (in brief by the formula which expresses the fact that the norm of the product of two numbers $a + b\theta$ and $c + d\theta$ of the field $R(\theta)$ equals the product of their norms), provided the integral algebraic numbers of the field $R(\theta)$ obey the laws of divisibility of arithmetic, and this condition is satisfied only for the following* 45 *values* of $m$ numerically $\leqq 100$, $m$ having no square factor:*

$m = -1, -2, -3, -7, -11, -19, -43, -67, 2, 3, 5,$

$6, 7, 11, 13, 14, 17, 19, 21, 22, 23, 29, 33, 37, 38, 41, 43, 46, 47,$

$53, 57, 59, 61, 62, 67, 69, 71, 73, 77, 83, 86, 89, 93, 94, 97.$

5. *Definition of Ideals. Known Theorems.* When the laws of divisibility fail for the integral algebraic numbers of a field, we may restore those laws by the introduction of ideals, as was first done by Kummer for a field defined by a root of unity, and by Dedekind for any algebraic field. By an *ideal* of a field $R(\theta)$ is meant any set $S$ of integral algebraic numbers of $R(\theta)$, not composed of zero only, such that the sum and difference of any two (equal or distinct) numbers of the set $S$ are themselves numbers of this set, while every product of a number of the set $S$ and an integral algebraic number of the field $R(\theta)$ is a number of the set $S$.

* The cases in which all ideals of the field are principal ideals, J. Sommer, *Zahlentheorie*, 1907, tables, pp. 346–358. Dickson, this BULLETIN, vol. 17 (1910–11), pp. 534–7, proved that if $m = -P$ is negative, 163 is the only value of $P$ between 67 and 1,500,000.

If $s$ ranges over the numbers of an ideal $S$, and $s_1$ ranges over the numbers of an ideal $S_1$, where $S$ and $S_1$ are ideals of the same field $R(\theta)$, then the products $ss_1$ and their linear combinations with rational integral coefficients form an ideal of $R(\theta)$, called the *product* of the factors $S$ and $S_1$, and denoted by $SS_1$ or by $S_1S$.

For quadratic fields, the case in which we are here interested, the theory of ideals has been developed quite simply.* Two notations are needed. First, $[k, l]$ denotes the totality of linear homogeneous functions of $k$ and $l$ with rational integral coefficients. Second, $\{k\}$ denotes a *principal ideal*, defined as the totality of the products of $k$ by integral algebraic numbers $x + y\theta$ of our field $R(\theta)$, where $x$ and $y$ are rational integers. Hence we have

$$\{k\} = [k, k\theta].$$

THEOREM 4. *In a quadratic field $R(\theta)$, where $\theta$ is defined by* (1) *or* (1′), *all ideals are given by $[ne, n(f + \theta)]$, where $n$, $e$, $f$ are rational integers such that*

(8) *or* (8′) $\quad f^2 \equiv m \pmod{e}, \quad f^2 + f + \tfrac{1}{4}(1 - m) \equiv 0 \pmod{e},$

*in the respective cases* (1) *or* (1′).

THEOREM 5. *In a quadratic field $R(\theta)$, the product of the ideal $[ne, n(f + \theta)]$ by its conjugate $[ne, n(f + \theta')]$ equals the principal ideal $\{n^2e\}$.*

The positive integer $n^2|e|$ is called the *norm* of $[ne, n(f + \theta)]$. The norm of any principal ideal $\{k\}$ is $|N(k)|$.

An ideal $S$ is said to be *divisible* by an ideal $T$ when there exists an ideal $Q$ of the same field such that $S = TQ$. An ideal, which is different from the principal ideal $\{1\}$ and is divisible by no ideal other than itself and $\{1\}$, is called a *prime ideal*.

THEOREM 6. *If a prime ideal divides $AB$, it divides $A$ or $B$. Every ideal which is neither $\{1\}$ nor a prime can be expressed in one and but one way as a product of a finite number of prime ideals. Hence ideals obey the laws of divisibility of arithmetic.*

Two ideals $A$ and $B$ are called *equivalent* if there exist principal ideals $\{\alpha\}$ and $\{\beta\}$ such that $\{\alpha\}A = \{\beta\}B$; we write $A \sim B$. If $A$ is equivalent also to $C$, with $\{\delta\}A = \{\gamma\}C$,

* Dickson, this BULLETIN, vol. 13 (1906–7), pp. 353–6; and, for a very detailed treatment of the case $m = -5$, ANNALS OF MATH., (2), vol. 18 (1917), pp. 169–178.

then $\{\beta\delta\}B = \{\alpha\gamma\}C$ and $B$ is equivalent to $C$. Hence all the ideals of a field which are equivalent to a given one are equivalent to each other, and are said to form a *class of ideals*. The *principal class* contains all the principal ideals and no others. For, if $A \sim \{1\}$, $\{\alpha\}A = \{\beta\}$, so that the number $\beta$ of the product is in $\{\alpha\}$, whence $\beta$ is divisible by $\alpha$, and $A = \{\beta/\alpha\}$.

6. *Application of Ideals to our Problem.* We now dispense with the assumption made in § 4 that the integral algebraic numbers of our quadratic field $R(\theta)$ obey the laws of divisibility of arithmetic. We shall examine by means of the theory of ideals our assumption that the solutions (5) of

$$N(x + y\theta) = zw \tag{9}$$

are all divisible by the prime $p$. The prime ideal $\{p\}$ therefore divides the product $\{z\}$ of the principal ideals $\{c + d\theta\}$ and $\{c + d\theta'\}$, without dividing either of the latter. Hence $\{p\}$ is not a prime ideal (§ 5). Thus $\{p\} = PP'$, where $P$ and $P'$ are conjugate prime ideals (which may coincide). By choice of the notation between $P$ and $P'$, we may assume that $P$ is the one dividing $\{c + d\theta\}$, i.e.

$$\{c + d\theta\} = PL, \tag{10}$$

where $L$ is an ideal of our field $R(\theta)$. By hypothesis, $p$ divides $x$ and $y$, whence, by (4), $\{p\} = PP'$ divides the product of the principal ideals $\{c + d\theta\}$ and $\{a + b\theta\}$. Since $\{c + d\theta\}$ is divisible by $P$, but not by $PP' = \{p\}$, it follows that $\{a + b\theta\}$ is divisible by $P'$:

$$\{a + b\theta\} = P'T, \tag{11}$$

where $T$ is an ideal of our field $R(\theta)$.

(*a*) First, let $L$ be a principal ideal. Then, by (10), $P$ is equivalent to the principal ideal $\{1\}$ and hence $P$ is a principal ideal. Evidently its conjugate $P'$ is a principal ideal. Then, by (11), $T$ is equivalent to, and hence equal to, a principal ideal. Hence equations (10) and (11) between principal ideals yield* equations (6) and (7) between numbers of our field, from which we conclude as in § 4 that the quotients of

* After inserting or removing unit factors, since $\{\pi\} = \{\lambda\}$ implies that $\lambda$ is the product of $\pi$ by a unit and conversely.

$x, y, z$, and $w$ by $p$ are the form (5) with $a, b, c$, and $d$ replaced by new integers $A, B, C$, and $D$.*

(*b*) Second, let $L$ be equivalent to an ideal $S$ which is not a principal ideal. If $S'$ is the conjugate to $S$, $SS' = \{e\}$, where $e$ is a rational integer (Theorem 5). Then we shall have $LS' \sim SS' = \{e\}$, so that $LS'$ is a principal ideal $\{\delta\}$. Multiplying (10) by $S'$, we get

$$S'\{c + d\theta\} = P\{\delta\}. \tag{12}$$

Multiplying (12) by $S$, and using $SS' = \{e\}$, we see that $SP\{\delta\}$ is a principal ideal, so that $SP \sim \{1\}$, whence $SP$ is a principal ideal. Its conjugate $S'P'$ is a principal ideal. But the product of (11) by $SS' = \{e\}$ shows that $S'P' \cdot ST$ is a principal ideal. Thus $ST \sim \{1\}$ and $ST$ is a principal ideal $\{\epsilon\}$. Hence, by (11),

$$S\{a + b\theta\} = P'\{\epsilon\}. \tag{13}$$

By the norms of the members of (12) and (13), in connection with (5), we get

$$|e\,z| = |eN(c + d\theta)| = p|N(\delta)|,$$
$$|ew| = |eN(a + b\theta)| = p|N(\epsilon)|.$$

By hypothesis, $x, y, z$, and $w$ are divisible by $p$. Hence $N(\delta)$ and $N(\epsilon)$ are divisible by $e$. By comparing the product of (12) and (13) with (4), we get

$$\{e\}\{x + y\theta\} = \{p\}\{\delta\epsilon\}.$$

Hence $e(x + y\theta)/p = \delta\epsilon u$, where $u$ is a unit. Replacing $\epsilon u$ by $\epsilon$, we conclude that the quotients of $x, y, z$, and $w$ by $p$ are integers $X, Y, Z$, and $W$, such that

$$X + Y\theta = \frac{\delta\epsilon}{e}, \qquad Z = \pm\frac{N(\delta)}{e}, \qquad W = \pm\frac{N(\epsilon)}{e}. \tag{14}$$

The requirement that $\delta\epsilon$, $N(\delta)$, $N(\epsilon)$ be divisible by $e$ may be expressed by congruential conditions modulo $e$ upon the four coordinates of $\delta$ and $\epsilon$ (cf. § 7). In the future we shall retain only the upper sign in (14) and understand that the

* If we attend only to the numbers $z$ of our solutions, we see that our discussion, with omission of the details leading to (11) and (13), leads to the quadratic forms of all divisors of the numbers represented by the quadratic form $N$.

simultaneous change of signs of $Z$ and $W$ in any solution leads to a companion solution which will not be listed. Our initial solution (5) is the case $e = 1$ of (14). The removal from (5) of a prime factor led us to the solutions (14). Bearing in mind that we must remove from (5) in succession the various prime factors of the common factor of the numbers (5), we may state the following lemma.

LEMMA. *From each class of ideals of the field $R(\theta)$, where $\theta$ is defined in terms of $m$ by* (1) *or* (1'), *select a representative ideal and call its norm $e$. For each $e$ impose on the coordinates of $\delta$ and $\epsilon$ the conditions that the divisions indicated in* (14) *are possible and derive the resulting solution* (14) *in integers. Similarly, examine the conditions that the four numbers in any such formula* (14) *shall be divisible by an arbitrary one of the numbers $e$ and derive the resulting solution in integers. Delete one of two such solutions if they are equivalent,* i.e. *if they differ only by a change of integral parameters. Repeat the process until closure results, so that the final sets of solutions $S_1, \cdots, S_k$ are such that, when the numbers of any $S_i$ are divisible by any $e$, the resulting solution is equivalent to one of $S_1, \cdots, S_k$. Then all integral solutions of $N(x + y\theta) = zw$ are integral multiples of $S_1, \cdots, S_k$.*

The theory of the correspondence* between classes of ideals and classes of quadratic forms and the theory of the composition of classes would seem to entitle us to pass from the preceding result to the following conjectured theorem.

THEOREM 7. *Select a representative $S = [e, f + \theta]$ of each class of ideals of the field $R(\sqrt{m})$, and define $\theta$ by* (1) *or* (1'). *Then all integral solutions of $N(x + y\theta) = zw$ are integral multiples of*

$$(15)\quad \begin{aligned} x &= eln + fnq - flr - gqr, & y &= lr + nq, \\ z &= el^2 + 2flq + gq^2, & w &= en^2 - 2fnr + gr^2, \\ m &\equiv 2 \text{ or } 3 \pmod 4, & f^2 - eg &= m, \end{aligned}$$

*or of*

$$(16)\quad \begin{aligned} x &= eln + fnq - (f+1)lr - gqr, & y &= lr + nq, \\ z &= el^2 + (2f+1)lq + gq^2, & w &= en^2 - (2f+1)nr + gr^2, \\ m &\equiv 1 \pmod 4, & (2f+1)^2 - 4eg &= m. \end{aligned}$$

* To $S$ and its conjugate $S'$ correspond $z$ and $w$, while to their product (a principal ideal) corresponds $N(x + y\theta)$, which therefore can be obtained from $z$ and $w$ by composition. This suggests another, but more technical, approach to our whole subject.

When $S$ is the principal ideal $\{1\}$, $e = 1, f = 0$, (15), and (16) with $n$ replaced by $n + r$, become $(5_1)$ and $(5_2)$ for $l = c$, $q = d$, $n = a$, $r = b$.

I have verified Theorem 7 for $m = -14, -17, -46$, when there are 4 classes of ideals; for $m = -26$, when there are 6 classes; for all values of $m$ numerically $< 100$ for which there are 2 or 3 classes (§ 7); and when there is a single class (§ 4). A general proof is being sought by one of my students, who is also applying the method to other types of Diophantine equations.

7. *Cases of 2 or 3 Classes of Ideals.* The following result, in connection with Theorem 3, disposes of all positive values of $m < 100$, except $m = 82$, in which case alone the number of classes of ideals exceeds 3.

THEOREM 8. *For the 37 values of $m$ between $-100$ and $+100$ and without a square factor for which there are exactly 2 or 3 classes of ideals in the field $R(\sqrt{m})$, all integral solutions of $N(x + y\theta) = zw$ are integral multiples of (5) and (15) or (16), where $e$ and $f$ take the one set or the two sets of values in one or two ideals $[e, f + \theta]$ which together with $\{1\}$ give representatives of the 2 or 3 classes of ideals.*

For $m \equiv 2$ or $3 \pmod 4$, $\theta = \sqrt{m}$, we write

$$\delta = D + q\theta, \qquad \epsilon = E + r\theta. \tag{17}$$

Then

$$N(\delta) = D^2 - mq^2, \qquad N(\epsilon) = E^2 - mr^2,$$
$$\delta\epsilon = DE + mqr + (Dr + Eq)\theta. \tag{18}$$

First, consider an ideal $S = [e, f + \theta]$ for which $e$ is a prime factor of $m$. By (8), $f \equiv 0 \pmod e$ and we may take $f = 0$. Note that every ideal, not a principal ideal, is equivalent to $S$ when * $m = -6, -10, -22, -58, 10, 26, 30, 42, 58, 70, 74, 78$ with $e = 2$, and when $m = 51$ or $66$ with $e = 3$. The numbers (18) are divisible by $e$ if and only if $D$ and $E$ are. For $D = el$, $E = en$, (14) become (15) with $f = 0$.

Second, for $m \equiv 3 \pmod 4$, let $S = [2, f + \theta]$. By (8), we may take $f = 1$. Note that every ideal, not a principal ideal, is equivalent to $S$ when $m = -5, -13, -37, 15, 35, 39, 55, 87, 91, 95$. Since (18) shall be divisible by 2, $D = q + 2l$, $E = -r + 2n$, and (14) with $e = 2$ become (15).

* Sommer, *Zahlentheorie*, 1907, pp. 346–358. (Tables.)

Third, for $m \equiv 2$ or $3 \pmod 4$, $m \equiv 1 \pmod 3$, let $S = [3, f + \theta]$. By (8), we may take $f = 1$ or $-1$. For $m = 31$ or 79, the three classes of ideals are represented by $\{1\}$ and the two $S$'s; for $m = 34$, the two classes are represented by $\{1\}$ and $S$ with $f = +1$. The numbers (18) are divisible by 3 if and only if $D \equiv \pm q, E \equiv \mp r \pmod 3$. For $D = \pm q + 3l$, $E = \mp r + 3n$, (14) with $e = 3$ become (15) with $f = \pm 1$.

These three main cases cover all the values of $m$ between $-100$ and $+100$ for which $m \equiv 2$ or $3 \pmod 4$ and for which there are exactly 2 or 3 classes of ideals.

It remains to verify closure. Let $x, \cdots, w$ in (15) be divisible by $e$. For our first case, $f = 0$ and $g = -m/e$ is not divisible by $e$, whence $q \equiv r \equiv 0 \pmod e$. The last is true also in our second case $e = 2, f = 1, g = (1 - m)/2$ odd. For $q = ed, r = eb$, the quotients of (15) by $e$ are of the form $(5_1)$ with $a = n - fb, c = l + fd$. In the third case $e = 3$, $f = \pm 1$, either $q \equiv r \equiv 0$ (just treated) or $l \equiv fgq, n \equiv -fgr \pmod 3$.

First, let $g \equiv -1 \pmod 3$ and hence replace $l$ by $-fq + 3l$, and $n$ by $fr + 3n$; the quotients of (15) by 3 are of the form (15) with $e$ replaced by 9, $f$ by $-2f$, and $g$ by $(g + 1)/3$. For $m = 31$, we apply to $z = (9, -2f, -3)$, with $f = \pm 1$, in succession the substitutions

$$\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}, \qquad \delta = \pm 2, \mp 1, \pm 1, \mp 5,$$

and obtain $(-3, \pm 4, 5)$, $(5, \pm 1, -6)$, $(-6, \pm 5, 1)$, $(1, 0, -31)$. The product of these substitutions is $l = 2c \mp 11d$, $q = \mp 5c + 28d$. Using also $n = 2a \pm 11b, r = \pm 5a + 28b$, we see that our solution becomes $(5_1)$.

Next, let $g \equiv +1 \pmod 3$ and hence replace $l$ by $fq + 3l$, and $n$ by $-fr + 3n$; the quotients of (15) by 3 are

$$\begin{aligned} x &= 9ln + 4fnq - 4flr - hqr, \qquad y = lr + nq, \\ (19) \qquad z &= 9l^2 + 8flq + hq^2, \qquad w = 9n^2 - 8fnr + hr^2, \\ h &= (g + 5)/3. \end{aligned}$$

For $m = 79$, apply to $z = (9, \pm 4, -7)$ in turn

$$\begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix}, \qquad \delta = \pm 1, \qquad \delta = \mp 1.$$

We obtain

$$z = 10s^2 \pm 14st - 3t^2 \equiv \phi \quad \text{for} \quad l = -s \mp t,$$

$q = \mp s - 2t$. But our initial $z$ in (15) with $e = 3, f = \pm 1$, $g = -10$, becomes $-\phi$ for $l = -t \pm 2s$, $q = s$. Eliminating $s$ and $t$, we see that if we replace $l$ by $\pm l - 3q$, $q$ by $2l \mp 5q$, $n$ by $\mp n - 3q$, and $r$ by $2n \pm 5r$ in (19), we obtain $x, -y, -z, -w$ of our initial (15). But by the change of the signs of $q$ and $r$ in (15), $y$ is changed in sign, while the effect on $x, z, w$ is to change the sign of $f$.

Finally, let $m = 34$, $e = 3$, $f = +1$. In (19) we take $l = -c + 6d$, $q = c - 5d$, $n = c + 6b$, $r = a + 5b$, and obtain $(5_1)$ with $z$ and $w$ changed in sign.

Let $m \equiv 1 \pmod 4$, $\theta = \frac{1}{2}(1 + \sqrt{m})$, and consider an ideal $S$ of the field $R(\theta)$ for which $e$ is a prime factor of $m$. By (8′), we may take $2f + 1 = e$. Note that every ideal, not a principal ideal, is equivalent to $S$ when $m = -35, 65, 85$ with $e = 5$; $m = -51$, $e = 3$; and $m = -9$, $e = 7$. By (2′) and (17),

$$N(\delta) = D^2 + Dq + \tfrac{1}{4}(1 - m)q^2,$$

$$N(\epsilon) = E^2 + Er + \tfrac{1}{4}(1 - m)r^2, \tag{20}$$

$$\delta\epsilon = DE + \tfrac{1}{4}(m - 1)qr + (Dr + Eq + rq)\theta.$$

Thus

$$4N(\delta) \equiv (2D + q)^2 \equiv 0, \qquad 4N(\epsilon) \equiv (2E + r)^2 \equiv 0 \pmod e.$$

Hence write $D = fq + el$, $E = -\frac{1}{2}(e + 1)r + en$. Then (14) give (16). To prove closure, let the numbers (16) be divisible by $e$. Since $m$ is not divisible by $e^2$, $g$ is not divisible by $e$. Hence $q \equiv r \equiv 0 \pmod e$. For $q = ed$, $r = eb$, the quotients of (16) by $e$ are of the form $(5_2)$ for $a = n - (f + 1)b$, $c = l + fd$.

Let $m \equiv 1 \pmod{4e}$, where $e$ is a prime. By (8′), we may take $f = 0$ or $f = -1$, and obtain conjugate ideals. The conditions that (20) be divisible by $e$ are

$$D(D + q) \equiv 0, \qquad E(E + r) \equiv 0, \qquad DE \equiv 0,$$

$$(D + q)(E + r) \equiv 0 \pmod e.$$

Hence either $D \equiv E + r \equiv 0$ or $E \equiv D + q \equiv 0$. For $D = el$,

$E = -r + en$, (14) become (16) with $f = 0$. For $E = en$, $D = -q + el$, (14) become (16) with $f = -1$. We may restrict the proof of closure to the first case. For, our two cases are interchanged by the substitution $s = (DE)(qr)(nl)$, which by (17) and (14) gives rise to the interchange of $\delta$ with $\epsilon$, and $z$ with $w$. But $s$ replaces $x, y, z, w$ of (16) with $f = 0$ by $x, y, w, z$ of (16) with $f = -1$. The numbers (16) with $f = 0$ are divisible by $e$ if and only if either $q \equiv r \equiv 0$ (treated above), or $l \equiv -gq$, $n \equiv gr \pmod{e}$.

First, let $g \equiv 0 \pmod{e}$. Replacing $l$ by $el$ and $n$ by $en$ in (16), we see that the quotients by $e$ are of the form (16) with $e$ replaced by $e^2$ and $g$ by $g/e$, while $f$ remains zero. For $m = -31$, $e = 2$, whence $g = 4$, we replace $l$ by $q$, $q$ by $-l$, $n$ by $-r$, and $r$ by $n$, and obtain (16) with $f = -1$. For $m = -15$, $e = 2$, we replace $l$ by $-d$, $q$ by $c + d$, $n$ by $b$, and $r$ by $-a$, and obtain $(5_2)$.

Next, let $g \equiv \pm 1 \pmod{e}$. Write $l = \mp q + eQ$, $n = \pm r + eR$. The quotients of (16) with $f = 0$ by $e$ are

$$x = e^2QR - (1 \mp e)rQ \mp eqR - \sigma qr, \quad y = Qr + Rq,$$

$$(21) \quad z = e^2Q^2 + (1 \mp 2e)Qq + \sigma q^2, \quad w = e^2R^2 - (1 \mp 2e)Rr + \sigma r^2,$$

$$\sigma = 1 + (g \mp 1)/e.$$

For $m = -23$, $e = 2$, we have $g = 3$ and may choose the upper signs. Replacing $Q$ by $-q$, $q$ by $l - q$, $R$ by $r$, and $r$ by $-n - r$, we obtain (16) with $f = -1$.

For $e = \sigma = 3$, the upper signs give $g = 7$, $m = -83$, and the lower signs give $g = 5$, $m = -59$. Replacing $Q$ by $q$, $q$ by $-l \pm q$, $R$ by $r$, and $r$ by $-n \mp r$, we obtain $-x, -y, z, w$ of (16) with $f = -1$.

These values $-15, -23, -31, -35, -51, -59, -83, -91, 65, 85$ are the only ones of $m$ between $-100$ and $+100$ which are $\equiv 1 \pmod 4$ and for which there are exactly 2 or 3 classes of ideals. Hence Theorem 8 is proved.

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# THE TRANSFORMATION OF ELLIPTIC INTEGRALS.

BY PROFESSOR J. H. MCDONALD.

1. *Introduction.* Jacobi discovered the transformation of the fifth order and proposed the problem of the transformation of order $n$. The solution of a system of algebraic equations is required. The number of arbitrary quantities is equal to the number of equations, but the direct solution could not be effected. The introduction of the inverse functions and the periods furnished a transcendental solution which cannot be regarded as complete till the transcendental elements are eliminated since the periods are not given. Cayley attempted an algebraic solution without success, as did also Clifford. Cayley says:

"The extension of this algebraic theory (Jacobi's determination of the transformations of degrees 3 and 5) to any value whatever of $n$ is a problem of great interest and difficulty: such theory should admit of being treated in a purely algebraical manner; but the difficulties are so great that it was found necessary to discuss it by means of the formulae of the transcendental theory, in particular by means of the expressions involving Jacobi's $q$ (the exponential of $-\pi k'/k$), $\cdots$. In the present memoir I carry on the theory algebraically as far as I am able; and I have, it appears to me, put the purely algebraical question in a clearer light than has hitherto been done; but I still find it necessary to resort to the transcendental theory."

In what follows the solution of Jacobi's system of equations is given independently of the transcendental theory and the foundation laid for a purely algebraic treatment of the whole subject of transformations.

2. *Jacobi's Problem.* Let $s$, $\Sigma$ be two conics such that it is possible to find a polygon of $n$ sides inscribed in $s$ and circumscribed about $\Sigma$. Then, by a well known theorem, there can be described an infinity of polygons having the same property. Let a parameter $t$ be introduced on the conic $s$; then the values of the parameters of the vertices of any polygon of the system are given by an equation $f(t, \lambda) = 0$ of degree $n$ in $t$ and of the first degree in $\lambda$.

To see this, let the equation of $s$ be $y^2 + xz = 0$ or $x = t^2$, $y = t$, $z = 1$. Then any quadratic equation $at^2 + bt + c \equiv a(t - t_1)(t - t_2) = 0$ may be regarded as determining the line joining the points whose parameters are $t_1$ and $t_2$. Any equation $f(t_1, t_2) = 0$ may be regarded as defining an envelope, viz. of those lines determined by values $t_1$, $t_2$ satisfying the equation. If the equation is of degree $n$ and symmetric in $t_1$ and $t_2$ the curve enveloped is of class $n$, if unsymmetric of class $n + m$ where $n$ and $m$ are the degrees in $t_1$, $t_2$. Let $A_0, \cdots, A_n$ represent $n + 1$ points on $s$; then the equation $\alpha_0/A_0 + \cdots + \alpha_n/A_n$ represents for different values of $\alpha_i$ curves tangent to the lines joining any two of the points $A_i$. This equation may be written in the form

$$\Sigma \frac{\alpha_i}{(t_1 - \alpha_i)(t_2 - \alpha_i)} = 0, \quad \text{or} \quad \Sigma \frac{\alpha_i}{t_1 - a_i} = \Sigma \frac{\alpha_i}{t_2 - a_i},$$

or

$$\frac{f(t_1) + \lambda\varphi(t_1)}{\varphi(t_1)} = \frac{f(t_2) + \lambda\varphi(t_2)}{\varphi(t_2)}.$$

This curve is tangent to the lines joining pairs of points given by $f(t) + \lambda\varphi(t) = 0$ and is of class $n$. It may be seen also that when a curve of class $n$ touches all lines joining pairs of a system of $n + 1$ points on a conic it touches the lines formed from an infinity of systems of $n + 1$ points on the conic.

Suppose a conic $\Sigma$ tangent to $n + 1$ of the lines, the constants $\alpha_i$ can be determined so that the curve of equation $\Sigma\, \alpha_i/A_i = 0$ touches $n$ additional tangents of $\Sigma$, or so that the curve and $\Sigma$ have $2n + 1$ tangents in common, or so that the curve must decompose and contain $\Sigma$ as one part together with a residue $\Gamma$. The curve $\Sigma\Gamma$ is tangent to the lines joining pairs of a succession of systems of $n + 1$ points on the conic $s$ and the conic $\Sigma$ is inscribed in a polygon of $n + 1$ sides connecting the points of each system. The curve $\Gamma$ also decomposes into conics or conics and a point when $n + 1$ is odd. For let $t_1, \cdots, t_{n+1}$ be the parameters of the vertices of any polygon circumscribed about $\Sigma$ and belonging to the systems and suppose them taken in order. Then the line joining $t_i$ to $t_{i+p}$ envelops a conic. For the relation between $t_i$ and $t_{i+p}$ is doubly quadratic, since to $t_i$ correspond $t_{i+p}$ and $t_{i-p}$ (the subscripts taken modulo $n + 1$). Hence this relation must be of the form

$$A(t_1^2 + t_2^2) + Bt_1t_2 + Ct_1t_2(t_1 + t_2) + D(t_1 + t_2) + Et_1^2t_2^2 + F = 0,$$

being necessarily symmetric. The envelope of the line must be of the second class. If $t_{i+p} = t_{i-p}$, the relation must be doubly linear and the envelope must be of the first class. This would occur if $n + 1 = 2m$ and $p = m$.

The polynomial whose roots are $t_1, \cdots, t_{n+1}$, as has been seen, must be of the form $f(t) + \lambda\varphi(t)$, or must belong to an involution. The double elements of the involution are $2n$ in number and are the parameters of points in which two vertices of a polygon coincide. These polygons are found by starting from a common point of $s$ and $\Sigma$ or from a point of contact of a common tangent. If $n + 1$ is odd, starting from a common point the polygon must consist of a succession of segments counted twice, and the tangent at the extremity, which must be a tangent of $\Sigma$. There are four such polygons, and the corresponding forms of the involution must be

$$f + \lambda_i\varphi = (t - \alpha_i)\psi_i^2 \qquad (i = 1, 2, 3, 4)$$

where $\alpha_i$ is the parameter of an intersection. If $n + 1$ is even, the singular polygons connect intersections or connect contact points of common tangents, and the forms of the involution are

$$f + \lambda_1\varphi = (t - \alpha_1)(t - \alpha_2)\psi_1^2,\ f + \lambda_3\varphi = (t - \beta_1^2)(t - \beta_2^2)\psi_3^2,$$

$$f + \lambda_2\varphi = (t - \alpha_3)(t - \alpha_4)\psi_2^2,\ f + \lambda_4\varphi = (t - \beta_3^2)(t - \beta_4)^2\psi_4^2,$$

where $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$ are the parameters of the points of contact. Since the double elements are properly accounted for, there are only four branch forms.

The forms $f$, $\varphi$ are seen to be subject to the same conditions as are the forms in Jacobi's problem. Conversely, if the involution $f + \lambda\varphi$ possesses branch forms of the same character as those that are required in Jacobi's problem, the forms $f$ and $\varphi$ lead to two conics in the poristic relation, and the involution curve decomposes. Suppose $n + 1$ odd and equal to $2m + 1$. Then to a polygon $(t - \alpha)\psi^2$ correspond $m(m - 1)/2$ double tangents of the involution curve, or $2m(m - 1)$ for the four forms. A curve of class $2m$ can only have this number if it decomposes. Every addition to the maximum number of double tangents for a proper curve must

be due to a further decomposition. For we have

$$1 + \frac{(p+q-1)(p+q-2)}{1} = \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq,$$

and since

$$2m(m-1) = \frac{(2m-1)(2m-2)}{2} + m - 1,$$

the involution curve must consist of $m$ parts. In the special case where the curve is an involution curve these must all be conics. For let $p_1, \cdots, p_m$ be the classes of the components, so that $p_1 + \cdots + p_m = 2m$; then some class must be equal to 2 or to 1. But if a point $p$ forms part of the involution curve, the order of the involution is even, since points on $s$ collinear with $P$ belong simultaneously to the involution. The class of some part must be 2 and a conic $\Sigma$ must be inscribed in lines of the system. If $\Sigma$ is tangent to fewer than $n+1$ lines, the forms $f$ and $\varphi$ must have a common factor, which is excluded by the assumption of a proper solution of Jacobi's problem. Then $\Sigma$ touches $n+1$ lines, and by the theory given above the involution curve completely decomposes into conics. If $n+1$ is even the conclusion is similar: the involution curve consists of conics and one point.

3. *Closure.* To effect the solution of Jacobi's system, it is necessary to consider the condition for closure. This is known under various forms. It is convenient to use a recurrence formula. Let there be two conics referred to the common self-polar triangle $(A)$ $x^2 + y^2 + z^2 = 0$, $(B)$ $ax^2 + by^2 + cz^2 = 0$. Take $M_0$ a point on $A$ and let $D_1$, the polar of $M_0$ with respect to $B$, meet $A$ in $M_1$, $M_{-1}$. The polar of $M_1$ meets $A$ in $M_0$ and a point $M_2$ and the polar of $M_{-1}$ meets $A$ in $M_0$ and $M_{-2}$. Let $D_2$ be the line $M_2M_{-2}$. In this way may be derived a series of points $M_pM_{-p}$ and lines $D_p$; the envelope of $D_p$ may be called $A_p$. From these definitions it follows, letting $M_0$ be $\xi\eta\zeta$, that the equation of $D_0$ is $\xi x + \eta y + \zeta\tau = 0$, that of $D_1$ is $a\xi x + b\eta y + c\zeta z = 0$, and the equation of $D_n$ is $a_n\xi x + b_n\eta y + c_n\zeta z = 0$, where $a_n$, $b_n$, and $c_n$ are functions of $a$, $b$, and $c$ independent of $\xi\eta\zeta$ and determined by the following relations of recurrence:

$$a_0 = b_0 = c_0 = 1, \qquad a_1 = a, \qquad b_1 = b, \qquad c_1 = c,$$

$$b_{2p} + c_{2p} = 2b_p{}^2c_p{}^2, \qquad cb_{2p-1} + bc_{2p-1} = 2b_{p-1}c_{p-1}b_pc_p,$$

$$c_{2p} + a_{2p} = 2c_p{}^2a_p{}^2, \qquad ac_{2p-1} + ca_{2p-1} = 2c_{p-1}a_{p-1}c_pa_p,$$

$$a_{2p} + b_{2p} = 2a_p{}^2b_p{}^2, \qquad ba_{2p-1} + ab_{2p-1} = 2a_{p-1}b_{p-1}a_pb_p.$$

There results the system of equations

$$\frac{b_n{}^2 - c_n{}^2}{b^2 - c^2} = \frac{c_n{}^2 - a_n{}^2}{c^2 - a^2} = \frac{a_n{}^2 - b_n{}^2}{a^2 - b^2},$$

$$\frac{b^2c_n{}^2 - c^2b_n{}^2}{b^2 - c^2} = \frac{c^2a_n{}^2 - a^2c_n{}^2}{c^2 - a^2} = \frac{a^2b_n{}^2 - b^2a_n{}^2}{a^2 - b^2},$$

from which follow

$$a_n{}^2 = G_na^2 + H_n, \qquad b_n{}^2 = G_nb^2 + H_n, \qquad c_n{}^2 = G_nc^2 + H_n,$$

where $G_n$ and $H_n$ are symmetric functions of $a^2$, $b^2$, and $c^2$, with $G_0 = 0$, $H_0 = 1$, $G_1 = 1$, $H_1 = 0$. They satisfy the equations

$$G_{p-1}G_{p+1} = H_p{}^2, \qquad G_{2p+1} = (G_pH_{p+1} - G_{p+1}H_p)^2,$$

$$G_{2p} = 4a_p{}^2b_p{}^2c_p{}^2G_p, \qquad G_2 = 4a^2b^2c^2.$$

Hence $G_n$ is the square of a symmetric function of degree $n^2 - 1$ in $a$, $b$, and $c$; i.e. $G_n = \Lambda_n{}^2$. The sign of $\Lambda_n$ is determined by the equations $\Lambda_2 = 2abc$, $\Lambda_{p-1}\Lambda_{p+1} = -H_p$.

The envelope of $D_n$ is

$$a_n{}^2x^2 + b_n{}^2y^2 + c_n{}^2z^2,$$

or

$$\Lambda_n{}^2(a^2x^2 + b^2y^2 + c^2z^2) - \Lambda_{n-1}\Lambda_{n+1}(x^2 + y^2 + z^2) = 0.$$

If $A_n$ coincides with $A_0$ the condition is $\Lambda_n = 0$. The condition that a polygon of $n$ sides inscribed in the conic $x^2 + y^2 + z^2 = 0$ is circumscribed about $a^2x^2 + b^2y^2 + c^2z^2 = 0$ is $\Lambda_n = 0$.

Involutions of the required character may be constructed from the equations of the lines $D_p$. Taking the cubic involution, we find that the condition for triangular closure is

$$(bc+ca+ab)(-bc+ca+ab)(bc-ca+ab)(bc+ca-ab)=0$$

or

$$b_2c_2 + c_2a_2 + a_2b_2 = 0.$$

If we set $\xi = 2t_0$, $\eta = 1 - t_0^2$, $\zeta = i(1 + t_0^2)$, the equation $f = 0$, where

$$f = (t - t_0)\{4a_2t_0t + b_2(1 - t_0^2)(1 - t^2) - c_2(1 + t_0^2)(1 + t^2)\},$$

gives the parameters of the points $M_0$, $M_2$, and $M_{-2}$. If we suppose $b_2c_2 + c_2a_2 + a_2b_2 = 0$, and let $t_0$ vary, the forms of a cubic involution are determined. In fact, the factor

$$4a_2t_0t + b_2(1 - t_0^2)(1 - t^2) - c_2(1 + t_0^2)(1 + t^2)$$

gives the affinity equation of the involution when equated to zero. The double elements are found by putting $t_0 = t$ and they are the roots of the equation

$$4a_2t^2 + b_2(1 - t^2)^2 - c_2(1 + t^2)^2 = 0.$$

Let us call these roots $\delta_1$, $\delta_2$, $\delta_3$, and $\delta_4$. The branch elements $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, and $\epsilon_4$ are the values of $t_0$ such that the affinity quadratic is a square. Hence they must satisfy the equation

$$(c_2^2 - b_2^2)t^4 + 2(a_2^2 + b_2^2 - 2a_2^2)t^2 + c_2^2 - b_2^2 = 0.$$

The equations for $\delta$ and $\epsilon$ are reciprocal equations involving only even powers of $t$, and it may easily be found that the branch forms of the involution are of the form

$$\lambda f + \mu\varphi = A_1(x + \epsilon)(x + \delta)^2, \qquad \mu f + \lambda\varphi = A_3\left(x + \frac{1}{\epsilon}\right)\left(x + \frac{1}{\delta}\right)^2,$$

$$\lambda f - \mu\varphi = A_2(x - \epsilon)(x - \delta)^2, \qquad \mu f + \lambda\varphi = A_4\left(x - \frac{1}{\epsilon}\right)\left(x - \frac{1}{\delta}\right)^2,$$

where $\delta$ is a root of the equation for the double elements and $\epsilon$ is the corresponding value of the branch element. It is easy to complete the solution of the transformation problem. It is seen that Cayley's normal form of the elliptic integral appears here.

If the order is 4, the involution is given by the equation

$$f = \{4at_0t + b(1 - t_0^2)(1 - t^2) - c(1 + t_0^2)(1 + t^2)\}\{4a_3t_0t + b_3(1 - t_0^2)(1 + t^2) - c_3(1 + t_0^2)(1 + t^2)\},$$

with $\Delta_4 = 4abca_2b_2c_2 = 0$, the roots of $f = 0$ being the parameters of the points $M_1$, $M_{-1}$, $M_3$, and $M_{-3}$. Suppose $a_2 = 0$, then $a_3 = -ab_2c_2$, $b_3 = bb_2c_2$, and $c_3 = cb_2c_2$; hence,

omitting the factor $b_2c_2$, the involution consists of the forms

$$b^2(1-t_0^2)^2(1-t^2)^2+c^2(1+t_0^2)^2(1+t^2)^2 - 2bc(1-t_0^4)(1-t^4)-16a^2t_0^2t^2,$$

with $a_2=-b^2c^2+c^2a^2+a^2b^2=0$, or

$$[(c-b)t_0^2+c+b]^2t^4+[(c+b)t_0^2+c-b]^2 + 2\frac{c^2-b^2}{c^2+b^2}[(c^2+b^2)t_0^4+2(c^2-b^2)t_0^2+c^2+b^2]t^2,$$

or, if we set $[(c-b)t_0^2+c+b]^2=\lambda$ and $[(c+b)t_0^2+c-b]^2=\mu$, of the forms

$$\lambda t^4+\frac{c^2-b^2}{c^2+b^2}(\lambda+\mu)t^2+\mu.$$

For the fifth order the affinity equation is

$$[4a_2t_0t+b_2(1-t_0^2)(1-t^2)-c_2(1+t_0^2)(1+t^2)] \times[4a_4t_0t-b_4(1-t_0^2)(1-t^2)-c_4(1+t_0^2)(1+t^2)]$$

with $\Lambda_5=0$. If we set $p=b_2c_2$, $q=c_2a_2$ and $r=a_2b_2$ it is found that

$$pqr\cdot\Lambda_5=(pq+qr-rp)(qr+rp-pq)(rp+pq-qr) - pqr(p+q+r)^3$$

and

$$a_4=q^2+r^2-p^2,\quad b_4=r^2+p^2-q^2,\quad c_4=p^2+q^2-r^2.$$

The discriminant of the involution is found by putting $t_0=t$ in the affinity equation. It must also be given by the resultant of the two factors, since $a$, $b$, and $c$ are subject to the relation $\Lambda_5=0$. If we develop the two forms of the discriminant and compare the coefficients, a number of forms of the closure condition result, but there are extraneous factors.

This method of constructing involutions is general and furnishes a complete solution of Jacobi's problem. The problem is always possible and determinate. It has been assumed that $a$, $b$, and $c$ are subject to no other relation than $\Lambda_n=0$. Such cases are of importance in the theory. Since the present purpose is to solve Jacobi's problem in its general form, they are left for further consideration.

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## BACHMANN ON FERMAT'S LAST THEOREM.

*Das Fermatproblem in seiner bisherigen Entwickelung.* By Paul Bachmann. Berlin and Leipzig, Walter de Gruyter, 1919. pp. viii + 160.

This volume reproduces to a considerable extent most of the important contributions which have so far been made toward a proof of Fermat's last theorem. It is far more complete than anything of the sort heretofore published. In particular, a reader of the book will find therein an account of the main results of Kummer, with proofs in most cases set forth in full. The writer wishes to call attention to the fact, however, that a number of references to articles bearing directly on some of the work given in the text have been omitted by Bachmann, a few of which will be noted, in detail, presently. If a better historical perspective is desired, it would be well for a reader to examine at the same time chapter 26, volume 2, of Dickson's *History of the Theory of Numbers.*

I shall now point out some parts of the text which give an account of results not given in detail elsewhere, aside from the original articles.* Consider

$$x^p + y^p + z^p = 0, \tag{1}$$

where $x$, $y$ and $z$ are rational integers, prime to each other, and $p$ is an odd prime. The assumption that $xyz$ is prime to $p$

* For an account of the more elementary results regarding the theorem, cf. Carmichael, *Diophantine Analysis*, chap. 5, or Bachmann, *Niedere Zahlentheorie*, vol. 2, chap. 9.

will be referred to as case I and the contrary assumption, $xyz \equiv 0 \pmod{p}$, as case II. In §§ 14–15 the researches of A. Fleck* are given. In §§ 23–24 the principles underlying Dickson's extensions of the method of Sophie Germain are set forth and the former's result is given that (1) is not satisfied in Case I for any $p < 7{,}000$ except perhaps 6,857.†

The proof of Dickson's theorem that there is at least one set of solutions in integers $x$, $y$ and $z$, prime to each other and the prime $q$, of

$$x^p + y^p + z^p \equiv 0 \pmod{q},$$

if

$$q \geqq (p-1)^2(p-2)^2 + 6p - 2,$$

is reproduced in full (§§ 25–26) as well as the proof by Schur of an analogous result.

In § 32, Bachmann begins the treatment of (1) by the use of the cyclotomic field theory following the methods of Kummer. The class number $h$ of the field defined by $e^{2i\pi/p}$ is referred to and the following statement (p. 104) is made:

"Die Kummersche Untersuchung ergibt ferner dass die Klassenzahl des Kreisteilungskörpers nur einmal durch eine Primzahl $p$ teilbar ist, wenn diese nur in einer der gennanten Bernoullischen Zahlen aufgeht . . . ."

The class number $h$ may have this property but Kummer's work does not prove it.‡

In §§ 36–37 is given substantially Hilbert's form of Kummer's proof that

$$\alpha^p + \beta^p + \gamma^p = 0$$

has no solution in integers $\alpha$, $\beta$, $\gamma$, belonging to the cyclotomic field defined by $e^{2i\pi/p}$, provided $p$ is a *regular* prime.

On page 111, the results given in Kummer's memoir§ of 1857 on Fermat's last theorem are mentioned, but no part of the argument is reproduced, except the derivation of the so-called Kummer criteria, namely that if (1) is satisfied in

* SITZUNGSBERICHTE MATH. GESELLSCHAFT BERLIN, vol. 8, p. 133, and vol. 9, p. 50.

† Dickson states that he has proved the result also for $p = 6857$, but he has not published the details. This would also follow from the relation $7 \times 6857 = 3 \times 2^7 \times 5^3 - 1$. See Vandiver, TRANS. AMER. MATH. SOC., vol. 15, p. 204.

‡ See the writer's criticism of a similar statement by Kummer, PROCEEDINGS NAT. ACAD. SCIENCES, vol. 6, p. 266 (May, 1920).

§ ABHANDLUNGEN, Berlin Academy, 1857, pp. 41–74.

Case I, then the congruences

$$B_s \frac{d_0^{p-2s} \log (x + e^v y)}{dv^{p-2s}} \equiv 0 \pmod{p} \qquad (s = 1, 2, \cdots, (p-3)/2), \tag{2}$$

all hold, where the $B$'s are the Bernoulli numbers, $B_1 = 1/6$, $B_2 = 1/30$, etc., and where the symbol $d_0^{p-2s}/dv^{p-2s}$ means that zero is substituted for $v$ after the differentiation is performed. Setting $x + e^v y = (x, y)$, then the same congruences hold with $(y, x)$, $(x, z)$, $(z, x)$, $(y, z)$ and $(z, y)$ substituted for $(x, y)$. The greater part of Kummer's proof of (2) is reproduced, but there are several points in the work which are not brought out by either Kummer or Bachmann, and which might puzzle a reader going over it for the first time. For example our author does not reproduce the argument employed by Kummer to establish the relation

$$\Pi_i (x + \alpha^{g^i} y) = \pm \alpha^m f(\alpha)^p, \tag{3}$$

where $g$ is a primitive root of $p$, and where $i$ ranges over the integers in the set $0, 1, 2, \cdots, p-2$, which have the property $g_{\mu-1} + g_{\mu-1+\text{ind } r} > p$, where $\mu = (p-1)/2$ and $g_s$ is defined as the least positive residue of $g^s$, modulo $p$, $r = 1, 2, \cdots, p-2$, $g^{\text{ind } r} \equiv r \pmod{p}$ and $\alpha = e^{2i\pi/p}$. In this connection it may be noted that the writer has not been able to justify Kummer's method* of showing that $\pm \alpha^m$ is the particular type of unit which appears in this relation.†

More details in the derivation of some of the other results regarding (3) would have been distinctly helpful to the reader; for example, in the derivation of relation 159 on page 114, and of the relation at the bottom of page 115.

On page 123, Bachmann outlines the method of Mirimanoff for proving that

$$\varphi(t) = (1+t)^{p-i} P_i(1, t) \qquad (i = 2, 3, \cdots, p-1)$$
$$\equiv t - 2^{i-1} t^2 + 3^{i-1} t^3 - \cdots - (p-1)^{i-1} t^{p-1} \pmod{p}$$

where

$$\frac{d_0^i \log (x + e^v y)}{dv^i} = \frac{P_i(x, y)}{(x+y)^i}.$$

* Kummer, loc. cit., p. 62, and CRELLE, vol. 35, p. 364.

† Compare with the writer's treatment of a similar problem, ANNALS OF MATHEMATICS, vol. 21, No. 2, Dec., 1919, pp. 74–75.

No reference is made to the much simpler method due to Frobenius* for obtaining the same result.

In connection with the derivation of the Kummer criteria, Bachmann does not mention (except on page 126 as to a minor detail) the researches of Cauchy, who anticipated Kummer in obtaining some of the important results related to these criteria. Cauchy† gave without proof a relation equivalent to (3) of this review for $r = 1$, and also stated that if (1) is possible in case I, then

$$1^{p-4} + 2^{p-4} + \cdots + ((p-1)/2)^{p-4} \equiv 0 \pmod{p},$$

which is a transformation of the criterion $B_{\mu-1} \equiv 0 \pmod{p}$. A consideration of these results in connection with the fact that Cauchy gave‡ a theorem regarding functions having properties similar to those of $\varphi(t)$, indicates the probability that he had obtained relations equivalent to two or more of the criteria of Kummer.

In §§ 47–48 an account is given of Frobenius' derivation§ of Mirimanoff's transformation of (2) which led the latter to the criteria $2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$, for the solution of (1) in case I. This work of Frobenius is based on the symbolic method of Blissard‖ for treating formulas involving Bernoulli's numbers.

On page 150, Bachmann makes a statement which would lead a reader to suppose that this work of Frobenius should be regarded as an introduction to the latter's later paper on Fermat's last theorem, the contents of which are not given by Bachmann. This is misleading, as the method employed in the second paper for deriving the relation

$$(x^p - 1)G_m{}^k(x) - (x^m - 1)F_{m;k}(x) \equiv H_m{}^{(k)}(x) \pmod{p}$$

is based on an extension of the method used by the writer¶ and is quite different from the method of Frobenius' earlier paper.

The book will constitute a valuable aid to anyone attempting a serious study of Fermat's last theorem.

H. S. VANDIVER.

* SITZUNGSBERICHTE MATH. GESELLSCHAFT BERLIN, July, 1910, p. 843.
† *Oeuvres*, (1), vol. 10, p. 362, Th. 3, and p. 364, Cor. 2.
‡ Loc. cit., p. 356, Th. 5.
§ SITZUNGSBERICHTE MATH. GESELLSCHAFT BERLIN, 1910, p. 200.
‖ QUARTERLY JOURNAL, vol. 4, 1861, p. 279. Erroneously attributed to Lucas by numerous writers.
¶ CRELLE, vol. 144, p. 314.

## TWO BOOKS ABOUT AIRPLANES.

*Aeronautics: A class text.* By Edwin B. Wilson. New York. John Wiley and Sons. vi + 265 pp.

*Grundlagen der Flugtechnik: Entwerfen und Berechnung von Flugzeugen.* Von Dr.-Ing. H. G. Bader. Berlin, B. G. Teubner, 1920. Mit 47 Figuren im Text. vi + 194 pp.

The mathematical treatment of the motion of an airplane requires a knowledge of two subjects, the dynamics of a rigid body, and the theory of fluid motion. The particular application is that of a rigid body moving in any way through a gas and supported by the reaction of the gas. In order to get this support, it is necessary that the body should have such a shape that the pressure of the gas may have full play, and that there be attached a power system which, by driving the body through the air, shall furnish a support equal to the weight of the body. If the body is required to rise, additional power is necessary to furnish an upward force to create the vertical acceleration. The first problem is, therefore, a consideration of the pressures to be produced when the body is to have uniform motion in a horizontal straight line, and next, the additional pressures when it is required to rise. Moreover, the best shape of the body for the purpose in view must be considered.

It is a familiar observation in watching the flight through the air of any object, such as a thin plate or a sheet of cardboard, that it rarely keeps in a fixed direction, or remains parallel to its first position, unless some device is used for the purpose. Even then there will be considerable deviations. In fact, one may almost say that, in general, the motion of a body through a fluid is unstable, and that, in order to keep it steady on a straight track, it is necessary to check the slight deviations as they arise. In other words, human agency is required for steering such an object. On the other hand, careful research, both mathematical and practical, have shown that we can often find devices which will maintain stability within certain limits. In the case of the airplane, investigation of the stability of any particular type is of fundamental importance, and this should extend not only to small deviations, but also to any that are likely to arise.

As far as the motion of the body is concerned when the various forces are given, the well known methods and results of ordinary dynamics can be applied immediately with full confidence. There are no fundamental difficulties to be overcome, except that of reducing complicated formulas to arithmetic. This is chiefly a matter of calculation, which is often laborious. The forces are of three kinds, those due to gravitation which can always be calculated, those due to the thrust of the power system for which we usually have the necessary data, and finally, those due to the air pressures. Theoretically, the latter should be deducible from the equations of hydrodynamics, but practically so little has been achieved in this effort, that in most cases recourse has been had to experiment both for the general laws and the detailed results.

It is really rather astonishing how little has been deduced for even the simplest case, say that of a sphere moving with constant velocity through a liquid. Solutions are known and are fully worked out for the case of a fluid without viscosity. But if the velocity exceeds a certain rather low limit, and the fluid has only a slight viscosity, the motion of the fluid and its pressures on the surface of the sphere bear little or no resemblance to those given by the solution for a non-viscous fluid. The motion is said to beome *turbulent*, which is merely a word to express our ignorance of its essential properties and our inability to calculate it. The mathematical solution for a perfect fluid cannot be treated as a first approximation to that in a slightly viscous fluid. The same fact vitiates nearly all the cases hitherto treated. This is made particularly obvious when we open the chapter on viscosity in a standard treatise, such as Lamb's *Hydrodynamics*, and find that in the very simplest problems the coefficient of viscosity appears in the denominator so that its vanishing makes the expressions infinite. Of course, there is an explanation of this result, but the explanation is after all really an apology for the inherent weakness of all the known methods of treatment for the motion of a solid through a liquid.

Thus, when we come to learn about the distribution of air pressures on a moving plane, we find that hydrodynamics helps but little. This is partly due to circumstances. During the war, when the greatest advance was made in the development of flying machines, it was necessary to obtain results as quickly as possible, and as there was little hope of getting them from

theory, experimental data were gathered by a numerous body of workers. It is from them that most of our present knowledge comes. But with calmer times, it is to be hoped that theory may take its proper share in the investigations and not only gather up into general formulas the known* numerical data, but also deduce further consequences and make advances on its own account.

Two or three volumes on the mathematics of the airplane have appeared in England. So far as we are aware, Professor Wilson's is the first to appear in this country. He is unusually well qualified for the task. Besides his eminence as a mathematician and a physicist, he has taught the subject for some years at the Massachusetts Institute of Technology, and has himself contributed a valuable memoir. Thus his views as to the proper approach for the student merit careful consideration. Like most of those who write introductory textbooks (and his is frankly of this character), he is naturally influenced by the character and degree of knowledge of the students who attend his own classes. This may cause difference of opinion as to what portions of previous subjects should be included. Aside from this fact, however, one can see well what is needed for the student approaching the subject of aeronautics, and the teacher can easily omit or add such matter as he may find necessary.

The first three chapters of Professor Wilson's text, constituting an introduction, give the principal facts which the student needs to know about the motion of a plane through the air. In the second chapter, the accepted law of pressure, —that it varies as the square of the velocity, the area of the surface, and the sine of the angle of attack (the angle between the plane and its direction of motion relative to the air),— is discussed, and the deviations from this statement for different velocities, shapes, and angles, as shown by experiment, are given and are illustrated by diagrams. With this as a basis, the equations of relative equilibrium of the skeleton plane are obtained in the third chapter, for steady horizontal, inclined, and circular flight. The effect of a second plane attached to the first as a *stabiliser* is also treated.

The next five chapters under the sub-title *Rigid Mechanics*, deal with the motion of the airplane under various conditions.

* The word *known* is perhaps inapplicable, for much of the information gathered by the various governmental stations is not yet available.

In the first of them, entitled *Motion in a resisting medium*, the landing problem, a free vertical fall and some other trajectories are discussed. In connection with the free fall, Professor Wilson uses an illustration showing how to construct an analytic expression which shall fit a table of numerical results derived from experiment. It may interest some readers to know that there *is* a maximum velocity of descent. Next follows a discussion of harmonic motion, which is not only sufficient for the purpose in view, but can be recommended as an excellent introduction to the subject apart from its present applications. Then follow three chapters which constitute the main treatment of the motion of an airplane from the point of view of a rigid body. The method is to form the general equations of motion, assume some form of steady motion which satisfies them, and then to determine the stability by finding the character of the small deviations from this steady motion, that is whether they are oscillatory, damped, or increase with the time. Professor Wilson introduces the student to the nature of the problem by solving some cases in two dimensions first and treating the general case in a later chapter. Those who have attempted to present Bryan's excellent but somewhat involved treatment of this subject to a class will welcome these chapters, which give the main portion of the work in sufficient detail for a good grasp of the methods to be obtained.

The third section, entitled *Fluid Mechanics*, deals partly with theoretical hydrodynamics, and partly with experimental results of observations in wind-tunnels and elsewhere. The author has tried to make the best of a bad job, but we fear that the student will recognize rather easily how little the theory assists in giving information about the air forces on an airplane. It is none the less necessary that he should learn at least the elements of fluid motion from the theoretical side, for one cannot doubt that the time will come when they will get fuller application.

A chapter on dimensions, besides having great importance for aerodynamics, on account of the necessary study of the behavior of models, fills a gap. Its use in determining the form of a functional relation between physical quantities when we know what variables are present, is occasionally hinted at but is not often adequately treated in a text-book.

Throughout the whole volume, numerical examples are used

as illustrations, most of them being taken from existing machines. In addition, Professor Wilson has gathered together at the end of each chapter numerous problems for the student to solve. If we regret anything, it is the lack of references to further work for the interested student or at least to show the rank and file they have not reached the conclusion of all knowledge on the subject. But this is a mere detail which will be filled in by the instructor who uses the book.

As its title denotes, the volume by Dr. Bader is devoted to the technique of the construction of airplanes, and as such it has no very special interest to the mathematician. But it is worth noting that the author is obviously well acquainted with the applications of mathematics to dynamical problems, both from the evidence of the book itself and from the titles of his own papers given in the list of the forty-three references at the end. He is full of his subject and he realizes that, in spite of basing all his results on theory, after all the final test must come from experiment. The maze of symbols used and the references to other work in the shape of formulas taken bodily with little explanation, repels the reader to some extent. If one searches for information as to the progress made in Germany in airplane construction, he will not find very much disclosed. Nevertheless, the aerodynamic expert need not consider the volume a useless addition to his library, for there are numerous hints here and there which will well repay some thought and examination. The copy received by the reviewer in paper covers fell to pieces on opening and is now a bundle of loose sheets. The price quoted on a slip enclosed with it is $1.30, which at the present rate of exchange, is equivalent to about 80 marks. This is probably about five times the price charged in the home market.

E. W. Brown.

## SHORTER NOTICES.

*Die Quadratur des Kreises.* By Eugen Beutel. Leipzig and Berlin, Teubner, 1920. 56 pp.

After stating clearly what is meant by a ruler and compass construction, the author gives a very complete history of this famous problem, which baffled mathematicians for twenty-five hundred years. He divides the history into three periods, the *geometric* period, the period of infinite series, the algebraic period. The first era, in which men tried to solve the problem by purely geometric methods, includes the work done by the Egyptians, Babylonians, Greeks, Romans, Hindus, Chinese, and the Christian nations up to the time of Newton. The work of Archimedes and of van Ceulen are the outstanding contributions in this period. When we stop to think that Archimedes was unfamiliar with the Arabic numerals and the decimal notation, we appreciate the magnitude of his task of computing the perimeter of the inscribed and circumscribed polygon of 96-sides in a circle of radius $R$. He found $3\frac{1}{7} < \pi < 3\frac{10}{71}$. Ludolf van Ceulen (1539–1610) carried Archimedes process of approximation still further and obtained the value of $\pi$ correct to 35 decimal places. It is in honor of this contribution that the Germans today often refer to $\pi$ as the Ludolfian number.

The second period contains the work of such men as Newton, Leibnitz, Gregorius von St. Vincentius, Kepler, Fermat and John Wallis. By means of infinite series these men were able to compute the value of $\pi$ to several hundred decimal places, but they were unable to solve the problem. However, their investigations led them to believe that the problem was incapable of solution.

It was then shown that if the problem is solvable, $\pi$ must not be transcendental. In 1873 Hermite proved that $e$ was transcendental. Making use of this fact and of Hermite's method of proof, Lindemann, in 1882 proved that $\pi$ was transcendental, settling forever the famous problem.

The book is well written and is well worth being read by every teacher of plane geometry.

FRANK M. MORGAN.

## NOTES.

The twenty eighth summer meeting of the American Mathematical Society will be held at Wellesley College, Wellesley, Massachusetts, on Wednesday, Thursday, and Friday, September 7–9. It will be preceded by the summer meeting of the Mathematical Association of America. A joint session of the two organizations has been arranged for the afternoon of Wednesday, September 7, with two speakers on phases of the Theory of Relativity. The usual joint dinner will be on Wednesday evening.

The regular sessions will be held in Founders' Hall. Tower Court will be open for the accommodation of members and friends, a separate wing being assigned to ladies and married couples. The Hall in this building will be available for social purposes during the meetings. General social functions and an excursion will be arranged.

Wellesley is fifteen miles from Boston, on the main line of the Boston and Albany Railroad. There is frequent train service from Boston as well as from Worcester and the West.

Titles and abstracts of papers intended for presentation at the meeting should be in the hands of the secretary by August 8, in order to appear on the printed program. Papers received after that date can be presented only as time permits. This announcement replaces the preliminary notice issued in the past. The attention of members is invited to the fact that no other preliminary notice will be issued.

The Division of Physical Sciences of the National Research Council was held in Washington on Wednesday, April 20. The following officers were elected for the ensuing year: chairman, Professor H. G. Gale; vice-chairman, professor E. P. Lewis; executive committee, Professors R. A. Millikan, Augustus Trowbridge, O. Veblen, John Zelleny. As members-at-large, Professors E. F. Nichols, S. W. Stratton, and G. W. Stewart were elected. The appointment of Professor E. B. van Vleck to succeed Professor H. S. White as one representative of the American Mathematical Society was announced. Professor L. E. Dickson reported for the American Branch of the International Mathematical Union that an invitation had

been extended by the American Branch at the Strasbourg meeting to hold the next International Congress of Mathematics in America, and that this invitation had been accepted. It is therefore presumed that such a Congress will be held in America in the year 1924 or as soon thereafter as is feasible. A similar report was made by Professor Dickson to the Council of the American Mathematical Society at its meeting on Friday evening, April 22, in New York.

In January, 1921, the National Association of German Mathematical Societies and Clubs was founded at Göttingen, the charter members including the German Mathematical Society, the Society for the Advancement of Mathematical and Scientific Education, the Mathematico-Scientific Society of Württemburg, and the mathematical societies of Berlin, Vienna, Hamburg, Göttingen, and Aachen. It is desired to unite in this association all mathematical societies of German speaking countries, whether their aim is primarily educational or scientific.

The newly founded Vereinigung der Freunde und Förderer des Positivistischen Idealismus announces in its organ, the ANNALEN DER PHILOSOPHIE, a prize of 5000 Marks for a paper on the following subject: *The relation of Einstein's relativity theory to modern philosophy, with special reference to the philosophy of "As-If."* Further information may be obtained from the editor of the ANNALEN DER PHILOSOPHIE, Dr. Raymond Schmidt, Fichtestrasse 13, Leipzig.

The American Association for the Advancement of Science has granted the sum of one hundred fifty dollars to Professor S. Lefschetz, of the University of Kansas, in support of his work in algebraic geometry.

Cambridge University has awarded Smith's prizes to L. A. Pars, of Jesus College, for an essay on *The general theory of relativity*, and to W. M. H. Greaves, for an essay on *Periodic orbits in the problem of three bodies.*

The Albert medal of the Royal Society of Arts has been conferred on Professor A. A. Michelson, of the University of Chicago, for his discovery of a natural constant which has provided a basis for a standard of length.

The University of Dublin has conferred honorary doctorates of science on Professor Emile Borel, of the Sorbonne, and Professor A. A. Michelson, of the University of Chicago.

Professor E. W. Brown, of Yale University, has been elected correspondent of the Paris Academy of Sciences in the section of astronomy, as successor to the late Professor E. C. Pickering, and Sir Ernest Rutherford has been elected correspondent in the section of physics, as successor to Professor A. A. Michelson, elected foreign associate.

At the University of Bonn, Associate Professor H. Beck has been promoted to a full professorship of mathematics. Professor H. Hahn has resigned, to accept a professorship of mathematics at the University of Vienna.

Professor L. Bieberbach, of the University of Frankfurt a. M., has been appointed successor to Professor C. Carathéodory at the University of Berlin.

Dr. H. Dingler and Dr. A. Rosenthal have been promoted to associate professorships at the University of Munich.

Professor E. Hellinger has been promoted to a full professorship of mathematics at the University of Frankfurt a. M.

Dr. L. Lichtenstein has been appointed honorary professor of mathematics at the Berlin Technical School.

At the Darmstadt Technical School, Dr. E. Naetsch has been appointed full professor of analytic geometry. Professor L. Henneberg has retired from active teaching.

Dr. J. Nielsen, of the University of Hamburg, has been appointed to a professorship at the Breslau Technical School.

Professor L. Prandtl, of the University of Göttingen, has been appointed professor of technical mechanics at the Munich Technical School.

Professor J. Thomae, of Jena, celebrated his eightieth birthday on December 11, 1920.

Dr. V. Geilen has been appointed lecturer in applied mathematics at the University of Münster.

Dr. W. Lorey has been appointed lecturer on the mathematics of insurance at the University of Leipzig.

Dr. W. Sternberg has been appointed privat docent in mathematics at the University of Heidelberg.

Dr. G. Pólya has been promoted to a professorship at the Zurich Polytechnikum.

Professor René Granier, of the University of Poitiers, has

at his own request been transferred from the chair of theoretical and applied mechanics to that of the differential and integral calculus.

Professor A. S. Eddington has been elected president of the Royal Astronomical Society.

The council of St. John's College, Cambridge, has appointed Dr. T. J. I'A. Bromwich prælector in mathematical science.

Associate Professor W. A. Manning has been promoted to a full professorship of mathematics at Stanford University.

Professor Frank Morley of Johns Hopkins University is now in England on leave of absence for the second semester of the current year.

Professor Albert Einstein will deliver a course of five lectures on relativity and gravitation theory at Princeton University on May 9–13. He also delivered lectures at Columbia University on April 16, and at the College of the City of New York on April 18–21.

Professor A. von Oettingen died at the age of eighty-four years.

The death is reported of Professor M. Petzold, of the Hanover Technical School.

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## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

Amodeo (F.). Le ricerche di un matematico napoletano del settecento su alcune teoremi di Archimede e sulle loro estensioni. Napoli, R. Accademia delle Scienze fisiche e matematiche, 1920.

Behrens (G.). Die Prinzipien der mathematischen Logik bei Schröder, Russell und König. (Diss., Kiel.) Hamburg, Druck von Berngrüber und Henning, 1918. 8vo. 65 pp.

Fawdry (R. C.). Coordinate geometry (plane and solid) for beginners. London, Bell, 1921. 8 + 215 pp. 5s.

Glause (H.). Geometrischer Beweis der Ergänzungssätze zum bikubischen Reziprozitätsgesetz. (Diss., Kiel.) Altona, Druck von Hammerich und Lesser, 1918. 8vo. 61 pp.

Hagström (K. G.). Der Begriff der statistischen Funktion. (Diss.) Uppsala, Almqvist och Wiksell, 1919. 8vo. 52 pp.

HEIBERG (J. L.). Naturwissenschaften, Mathematik und Medizin im klassischen Altertum. 2te Auflage. (Aus Natur und Geisteswelt, Nr. 370.) Leipzig, Teubner, 1920. 104 pp.

HOBSON (E. W.). The theory of functions of a real variable and the theory of Fourier's series. 2d edition. Volume 1. Cambridge, University Press, 1921. 16 + 671 pp. 45s.

KLEIN (L.). Streifzüge in das Gebiet der Mathematik und Geometrie. Korneuberg, Fellner und Zausner, 1920.

PALMQUIST (R.). Quelques études sur la convergence des déterminants infinis et les systèmes d'une infinité d'équations linéaires à une infini d'inconnus. (Diss.) Uppsala, Almqvist och Wiksell, 1915. 8vo. 52 pp.

PERL (A.). Ueber die singulären Punkte der algebraischen Flächen dritter Ordnung. Leipzig, Teubner, 1915. 8vo. 46 pp.

PRÜSS (A.). Ueber vollständig zerfallende algebraische Raumkurven und Flächen. (Diss., Kiel.) Kiel, Schmidt und Klaunig, 1918. 8vo. 55 pp.

REIMERS (M. H.). Zur Theorie der algebraischen Raumkurven und ihrer Tangentenflächen. (Diss., Kiel.) Leipzig, Teubner, 1915. 8vo. 55 pp.

SIMPSON (T. McN.). Relations between the metric and projective theories of space curves. (Diss., Chicago.) Private edition, distributed by the University of Chicago Libraries, 1920. 4to. 26 pp.

DE VRIES (H.). Leerboek der differentiaal- en integraalrekening en van de theorie der differentiaalverglijkingen. 2 deele. Groningen, Noordhoff, 1919–1920. 750 + 460 pp. f. 19.20 + 16.50

## II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Aufgabensammlung, methodisch geordnet. Neue Ausgabe nach F. Pietzker und O. Presler bearbeitet von A. Mohrmann. 10te Auflage. Leipzig, Teubner, 1920. 450 pp.

BOJKO (J.). Lehrbuch der Rechenvorteile. Schnellrechnen und Rechenkunst. (Aus Natur und Geisteswelt, Nr. 739.) Leipzig, Teubner, 1920. 115 pp.

DIECK (W.). Mathematisches Lesebuch. 1ter und 2ter Band. Sterkrade, Osterkamp, 1920.

DURELL (C. V.). A concise geometry. London, Bell, 1921. 8 + 319 pp. 5s.

GOLLER (A.). See Lengauer (J.).

GONGGRIJP (B.). Zaktafels in vier decimalen. Groningen, Noordhoff, 1919. 52 pp.

VAN DE GRIEND (J.). Trigonometrische vraagstukken met beknopte theorie. Antwoorden op de Trigonometrische vraagstukken met beknopte theorie. Groningen, Noordhoff, 1920. 8vo. 132 + 24 pp. f. 1.90 + 1.00

LENGAUER (J.). Die Grundlehren der ebenen Geometrie. 20te Auflage, von A. Goller. Kempten, Kösel, 1920. 160 pp.

MOHRMANN (A.). See BARDEY (E.).

PIETZKER (F.). See BARDEY (E.).

PRESLER (O.). See BARDEY (E.).

VAN THIJN (A.). Leerboek der vlakke meetkunde met opgaven. Eerste deel, vijfte druk. Groningen, J. B. Wolters, 1920. 8vo. 119 pp. f. 2.40

WEBER (M.). Les mathématiques de l'élève ingénieur. I: Algèbre et analyse. 1re partie: Notions fondamentales. Trigonométrie. Algèbre. Paris, Dunod, 1920. 8vo. 14 + 495 pp. Fr. 45.00

## III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. 2ter Band: Elektromagnetische Theorie der Strahlung. 4te Auflage. Leipzig, Teubner, 1920.

D'ADHÉMAR (R.). Résistance des matériaux. Paris, Gauthier-Villars, 1921. 8vo. 9 + 185 pp. Fr. 20.00

BOLZMANN (L.). See BUCHHOLZ (H.).

BORN (M.). Die Relativitätstheorie Einsteins und ihre physikalischen Grundlagen. Gemeinverständlich dargestellt. (Naturwissenschaftlichen Monographien und Lehrbücher, Band 3.) Berlin, Springer, 1920.

BOSE (S. N.). See EINSTEIN (A.).

BOUASSE (H.). Statique. Machines simples, bascules et balances, frottement, graissage, statique graphique. Paris, Delagrave, 1920. 8vo. 24 + 515 pp.

BRYANT (W. W.). Kepler. (Pioneers of Progress: Men of Science Series.) London, Society for the Promotion of Christian Knowledge, and New York, Macmillan, 1921. 12mo. 62 pp. 2s.

BUCHHOLZ (H.). Ludwig Boltzmanns Vorlesungen über die Prinzipe der Mechanik. 3ter Teil: Elastizitätstheorie und Hydromechanik. Leipzig, Barth, 1920. 820 pp.

CASE (J.). Theory of direct current dynamos and motors. Cambridge, Heffer, 1921. 13 + 196 pp. 15s.

CLAUZEL (G.). Effets gyroscopiques et méthodes vectorielles. Paris, Dunod, 1920. 8vo. 147 pp. Fr. 14.00

DISPER (P.). Ueber die Massenverteilung und Verschiebung der Druck- und Zugkräfte in einem Kometen. Montabaux, Kalb, 1919. 42 pp.

EGERER (A.). Kartenkunde. I: Einführung in das Kartenverständnis. (Aus Natur und Geisteswelt, Nr. 610.) Leipzig, Teubner, 1920. 146 pp.

EINSTEIN (A.) and MINKOWSKI (H.). The principle of relativity. Original papers by A. Einstein and H. Minkowski, translated into English by M. N. Saha and S. N. Bose. Calcutta, University Press, 1921. 23 + 186 pp.

ENGLUND (E.). Sur les méthodes d'intégration de Lie et les problèmes de mécanique céleste. (Diss.) Uppsala, Almqvist och Wiksell, 1916. 8vo. 50 pp.